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Generating high-quality high-order parameterization for isogeometric analysis on triangulations

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Abstract

This paper presents an approach for automatically generating high-quality high-order parameterizations for isogeometric analysis on triangulations. A B-spline represented boundary geometry is parameterized into a collection of high-order Bézier triangles or tetrahedra in 2D and 3D spaces, respectively. Triangular Bézier splines are used to represent both the geometry and physical fields over the triangulation. By imposing continuity constraints on the Bézier ordinates of the elements, a set of global C^r smooth basis is constructed and used as the basis for analysis. To ensure high quality of the parameterization, both the parametric and physical meshes are optimized to reduce the shape distortion of the high-order elements relative to well-defined reference elements. The shape distortion is defined based on the Jacobian of the triangular Bézier splines, and its sensitivity with respect to the location of control points is derived analytically and evaluated efficiently. Moreover, a sufficient condition is derived to guarantee the generated mesh is free of local self-intersection, thanks to the convex hull property of triangular Bézier splines. By using a Heaviside projection function, the non-negative Jacobian determinant constraints are formulated efficiently as a single optimization constraint. Several 2D and 3D numerical examples are presented to demonstrate that high-quality high-order elements are generated using our approach.

Keywords: Domain parameterization; Isogeometric analysis; Triangular Bézier splines; High-order elements; C^r smoothness

1. Introduction

Isogeometric analysis (IGA) is a numerical analysis approach introduced by Hughes et al. [1] to integrate Computer-Aided Design (CAD) and Finite Element Analysis (FEA). It uses the same Non-Uniform Rational B-Splines (NURBS) as basis to both represent the geometry and approximate field variables in solving partial differential equations (PDEs). Due to the same basis used in geometric representation and in solution approximation, it eliminates the geometric approximation error commonly occurred in classical FEA procedures. Once the initial mesh

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is constructed, refinements can also be easily implemented and exact geometry is maintained at all levels without the necessity of interaction with the CAD system [1,2]. The exact geometry representation has also led to the development of isogeometric shape optimization [3–5], where the optimized geometry can be directly imported into CAD systems. Another advantage of isogeometric analysis is its computational efficiency on a per-node basis over classical C^0 Lagrange polynomial based FEA. The higher continuity of the NURBS basis has been demonstrated to significantly improve the numerical efficiency and accuracy on a per-node basis in many areas including structural analysis [2,6], fluid simulation [7] and shape optimization [3–5,8].

However, the tensor-product structure of NURBS has restricted its applicability in analysis. First, many CAD models cannot be represented by one single NURBS patch. Therefore, multiple patches have to be used for geometry with complex topology and it is not easy to achieve high-order continuity between such patches. Besides, it is challenging to construct NURBS based volumetric mesh from surface representation of complex geometries [9–11]. Analysis of trimmed geometries [12] and local mesh refinement are also known to be cumbersome for NURBS based IGA.

As an alternative, triangular Bézier splines (TBS) have recently emerged as a powerful alternative to shape modeling and isogeometric analysis, due to their flexibility in representing domains of complex topology and their high-order of continuity. Local refinement can also be implemented without any difficulty. Normalized basis of Powell–Sabin (PS) splines has been used for numerical solution of PDEs in [13,14]. More generalized framework of IGA on triangulations is introduced in [15–17], where C^r smooth rational triangular Bézier splines (rTBS) are used as basis to represent both the geometry and physical fields. The rTBS elements can be locally refined and represent any geometric model of complex topology, including trimmed geometries [17]. By using a smooth-refine-smooth scheme, optimal convergence has also been demonstrated with C^r elements for generalized geometries [16,17]. The rTBS based IGA has also been used in shape optimization where optimized designs with complex topology have been demonstrated and can be directly linked to CAD systems [18]. Recently, an isogeometric approach based on unstructured tetrahedral and mixed-element Bernstein–Bézier elements has also been proposed [19].

Although high-order elements in IGA have exhibited high efficiency in analysis, parameterization quality is a general concern. A good parameterization requires the physical mesh to be valid, that is free of local self-intersection or folding. Although NURBS based IGA has been demonstrated to be robust with severe mesh distortion [20], a good parameterization can be beneficial to both the analysis accuracy and computation efficiency [21,22,5,11,23]. Both planar and volumetric B-spline based parameterization [5,22,11] have been investigated previously, where Jacobian based measurements are used as objectives to improve the parameterization quality through an optimization procedure. T-spline based parameterization has also been studied [24,25]. Parameterization in IGA based on PS splines is studied in [23], where the planar parameterization is improved by minimizing the Winslow functional. Generating high-quality triangular and tetrahedral Lagrange finite elements has also been studied in [26–28], where a shape distortion measurement is used as objective to improve the mesh quality. The shape distortion is calculated based on the Jacobian of the parameterization evaluated at sampled points in the elements.

In this work, we focus on generating high-quality high-order Bézier triangular and tetrahedral elements in IGA based on triangular Bézier splines [15–17]. Given a B-spline represented boundary geometry, we first generate a linear triangular/tetrahedral mesh over a polygonal domain that approximates the original geometry. Then we elevate the degree of the linear mesh to produce the initial parametric mesh. By replacing the boundary control points of the parametric mesh with the Bézier control points extracted from the input B-spline boundary, an initial high-order C^0 physical mesh is created. However, the replacement of boundary control points may result in self-intersection of elements near curved boundaries. Moreover, to construct C^r parameterization, some points in the parametric and physical meshes may need to be relocated to satisfy the high-order continuity constraints, and likely resulting in poor quality or even tangled elements. To untangle the meshes and improve the parameterization quality, we develop an approach to sequentially optimize the parametric and physical meshes with the goal to reduce the shape distortion of the elements relative to well-defined elements. In addition, we derive a sufficient condition to guarantee the generated elements to be free of local self-intersection, taking advantage of the convex hull property of the triangular Bézier splines. Directly incorporating the aforementioned sufficient condition for each element as constraints into the optimization would dramatically slow down the optimization, due to the large number of constraints. Instead we use a Heaviside projection based constraint formulation [29] to cast the large number of constraints into a single constraint, which significantly improves the optimization efficiency.

The remainder of this paper is organized as follows. Section 2 gives a brief introduction of bivariate and trivariate splines on Bézier triangles and tetrahedra respectively. Section 3 describes the process of constructing an initial