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Robust isogeometric preconditioners for the Stokes system based on the Fast Diagonalization method

M. Montardini[a], G. Sangalli[a,b], M. Tani[a,*]

[a] *Dipartimento di Matematica, Università degli Studi di Pavia, Italy*
[b] *Istituto di Matematica Applicata e Tecnologie Informatiche "E. Magenes" del CNR, Pavia, Italy*

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Abstract

In this paper we propose a new class of preconditioners for the isogeometric discretization of the Stokes system. Their application involves the solution of a Sylvester-like equation, which can be done efficiently thanks to the Fast Diagonalization method. These preconditioners are robust with respect to both the spline degree and mesh size. By incorporating information on the geometry parametrization and equation coefficients, we maintain efficiency on non-trivial computational domains and for variable kinematic viscosity. In our numerical tests we compare to a standard approach, showing that the overall iterative solver based on our preconditioners is significantly faster.

Keywords: Isogeometric analysis; k-method; Preconditioning; Stokes system; Kronecker product

1. Introduction

Isogeometric analysis (IGA) has been introduced by T.J.R. Hughes et al. in the seminal paper [1]. IGA is an innovative numerical method to discretize partial differential equations (PDEs), based on using the same functions that describe the computational domain in computer-aided design (CAD) systems also for the representation of the solution. These functions are B-Splines or NURBS or generalizations of them. For a complete description of the method and an overview of various engineering applications, see [2]. For a mathematical-oriented overview of IGA we refer to [3].

IGA is a high-order numerical method, when high-degree polynomial/spline approximation is adopted. However within IGA there is the possibility of high-regularity approximating functions. The typical case is indeed when splines of degree p and global C^{p-1} regularity are used within each patch. This is called the isogeometric k-method, which presents significant advantages in comparison to C^0 finite elements of degree p, from many points of view: higher

* Corresponding author.
E-mail addresses: monica.montardini01@universitadipavia.it (M. Montardini), giancarlo.sangalli@unipv.it (G. Sangalli), mattia.tani@unipv.it (M. Tani).

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accuracy per degree-of-freedom (see [4,3]), improved spectral behaviour (see [5]), the possibility of dealing directly with higher-order PDEs ([6] is the first paper in this direction) or constructing smooth structure-preserving schemes (see [7]).

In this paper the problem of interest is the Stokes system. We consider in particular two well-known isogeometric discretizations for which stability and convergence is known. One is the extension of the Taylor–Hood element, which is *inf–sup* stable, see [8,7,9–11]. The other is the extension of the Raviart–Thomas element, which is stable and structure-preserving, in the sense that the discrete solution is pointwise divergence-free; see [9,12] (and [13,14] for its extension to Navier–Stokes). Both allow for arbitrary degree and regularity, in the spirit of the k-method.

The k-method is not costless: the computational cost per degree-of-freedom when dealing with the k-method linear system grows as the degree and regularity increase. In this paper we focus on the cost of solving the system, which is only one part of the problem (the other important part is the formation of the system matrix, which is also an active research field). Linear solvers that are developed for finite elements (e.g., direct [15], iterative multilevel [16]) work well for low-degree isogeometric analysis but the computational performance deteriorates for the high-degree k-method. Recently, papers have appeared with preconditioners that behave robustly for the isogeometric k-method: [17] adopts a domain-decomposition approach, [18] and [19] are based on the multigrid idea (in particular, the latter contains a proof of robustness, based on the theory of [19]), and finally [20], which uses a direct solver at the preconditioner stage, and takes advantage of the tensor-product structure of the multivariate spline spaces. All these papers deal with the Poisson problem.

Isogeometric preconditioners for the Stokes system have also been studied in recent papers: [21,22] consider block-diagonal and block-triangular preconditioners combined to black-box solvers (either algebraic-multigrid or incomplete factorization); [23] studies the domain-decomposition FETI-DP strategy; [24] focuses on a multigrid strategy; another multigrid approach, which extends the results of [19], can be found in [25].

In the present work, for both Taylor–Hood and Raviart–Thomas isogeometric discretizations of the Stokes system, we consider preconditioners having the classical block structure (see [26]) and using direct solvers to invert the diagonal blocks.

In the simplest approach, our pressure Schur complement preconditioner is the pressure mass matrix in parametric coordinates, which is solved by exploiting its Kronecker structure. Moreover, our preconditioner for the velocity blocks is a component-wise Laplacian in parametric coordinates, and its solution is the solution of a Sylvester-like equation. The latter equation is well studied in the numerical linear algebra community (see for example the overview [27]); among many methods, following [20] we adopt a direct solver named Fast Diagonalization (FD) method, see [28,29].

An important problem we have to face is the treatment of the geometry parametrization. The simplest approach outlined above does not incorporate any geometry information in the preconditioner, causing a significant loss of efficiency on complex geometry parametrizations. To overcome this limitation, we propose a modification of the preconditioner for a partial inclusion of the geometry information, without increasing its computational cost. Even though the mathematical analysis of this modification is postponed to a later work, in our numerical benchmarking we show the clear benefits of this approach. Indeed, we show theoretically and numerically that our preconditioner is robust with respect to the mesh size h and spline degree p, both for the isogeometric Taylor–Hood and Raviart–Thomas methods. While previous papers considered low-degree splines only (typically quadratics and cubics), we are motivated to consider higher degrees in our tests (up to degree 6 for the velocity and 5 for the pressure, for memory constraints) by the fact that the computational cost of our preconditioner is almost independent of the degree. The iterative solver total computational time is $O(n_{dof}\, p^3)$, but it is heavily dominated by the matrix–vector multiplication which takes more than the 99% of the overall cost when the pressure degree is 5 and the velocity degree is 6, on a 16^3 elements mesh. In this case our preconditioners are much faster than the alternatives known in literature: for example, about 3 orders of magnitude when comparing to a standard preconditioner based on the incomplete Cholesky factorization, which is known to be an effective choice (see, e.g., [21]).

In conclusion our numerical benchmarks confirm that the proposed preconditioner is very efficient and well suited for the k-method. Further advances in the solver performance can be achieved with a matrix-free approach, that accelerates the matrix–vector multiplication operation, for moderate or large degree. A first step in this research direction is [30].

The outline of the paper is as follows. In Section 2 we give a short review of the Taylor–Hood and Raviart–Thomas isogeometric discretizations for the Stokes system, and summarize the main properties of the Kronecker product. The