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An energy-based discontinuous Galerkin discretization of the elastic wave equation in second order form

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Abstract

We present an application of our general formulation (Appelö and Hagstrom (2015) [12]) to construct energy based, arbitrary order accurate, discontinuous Galerkin spatial discretizations of the linear elastic wave equation. The resulting methods are stable and, depending on the choice of numerical flux, conserve or dissipate the elastic energy. The performance of the method is demonstrated for problems with manufactured and exact solutions. Applications to more realistic problems are also presented. Implementations of the methods are freely available at Appelö and Hagstrom (2015) [19].

Keywords: Discontinuous Galerkin; Elastic waves; Energy flux

1. Introduction

Accurate propagation of linear elastic waves in heterogeneous material and complex geometry is important in many fields such as design of mechanical structures, non-destructive testing, civil engineering and seismic exploration. In general, the challenge in numerically propagating waves is the long distance of propagation, requiring high order methods able to control the growth of dispersion errors.

Discontinuous Galerkin (dG) methods have excellent dispersion properties. The research devoted to dG has been substantial over the last decade and a half and we will not attempt to review the whole literature here (the textbook by Hesthaven and Warburton, [1], is an excellent reference) but limit our discussion to methods pertaining to the linear time dependent elastic wave equation.

Discontinuous Galerkin methods for elastic waves either discretize the governing equations in some first order formulation (in space and time), e.g. velocity–strain or velocity–stress, or directly work with the equations in second order form. Examples of methods based on first order systems which construct the numerical fluxes based on the Riemann problem are [2–4]. Another method that works with a first order system is the staggered dG method in [5].

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Methods that work with the equations in second order form are the interior penalty method [6] and the symmetric interior penalty method [7], as well as the local discontinuous Galerkin methods; see [8] for a stability analysis applicable to a wide range of methods of this type. Another class of methods is the so-called space–time discontinuous Galerkin methods; see for example the early application in [9] and more recently [10,11]. We postpone to Section 4 comparisons with other discretization techniques after our method, which differs from all of these, has been fully described.

In our proposed approach we approximate the equations in second order form in space but introduce the velocity to reduce the order to one in time. As the velocity is naturally connected to the kinetic energy and the displacement, through the stress, is connected to the potential energy, this form allows for a dG formulation that mimics the dynamics of the energy of the system. The proposed method is a direct application of our general formulation for wave equations in second order form [12]. The crucial (and to our knowledge novel) step in our formulation is to test the equation governing the time derivatives of the displacements, not directly against a test function but against a quantity related to the potential energy. This immediately leads to an energy identity for the dG formulation under very general assumptions on the approximation spaces. This test quantity is often invariant to certain transformations and therefore does not fully specify the time derivatives of the displacements. However we show that it is straightforward to add independent equations to close the system without affecting the energy identity.

Finite difference methods are also often used for wave propagation problems. Until recently an obstacle to constructing high order and curvilinear solvers for the second order formulation was the stable enforcement of traction boundary conditions, but there are now multiple options, [13–15], all relying on summation by parts techniques to prove energy stability. Although these methods typically have a smaller spectral radius than dG and spectral element methods, they may become less efficient when solving problems where free surface or interface waves are important.

The classic parameter used to measure resolution requirements for finite difference methods, points per wavelength (PPW) introduced in the seminal paper [16] by Kreiss and Oliger, has recently been found to be an inadequate measure of the resolution requirements when surface waves or interfaces are present [13,17]. The analysis of Kreiss and Petersson, [17], uses a modified equation approach to show that the number of points required to reach a fixed error at a fixed time scales with $(\lambda/\mu)^{1/p}h^p$ if a method that approximates the surface waves to pth order of accuracy is used. The results presented in the experiments section indicates that *this effect is not important for the proposed discretization* (and probably not for other dG methods either). As the analysis in [17] is based on a modified equation it does apply to dG as well, however it is well known that the wave speeds in dG discretizations are approximated with about twice the order of the method [7,18]. Now the dispersion error in the surface wave for dG is, say, $(\lambda/\mu)^{\frac{1}{2p}}h^{2p}$ which is still very small compared to h^p unless λ/μ is (unphysically) large. We believe that this is the reason why we do not observe any degradation when $\lambda/\mu \gg 1$.[1]

As the basic theoretical analysis and framework of the general method has already been presented in [12] we focus here on the formulation for the elastic wave equation and the method's performance for a sequence of numerical examples. Through these experiments, whose computer implementations are freely available from [19], we hope to convince practitioners that our method is worth considering. Here we work in two dimensions but note that the method can easily be generalized to three dimensions.

The rest of the paper is organized as follows. In Section 2 we present the governing equations and discuss the dynamics of the energy of a solid body subject to boundary conditions. In Section 3 we present the variational formulation and discuss how to add independent equations to compensate for the invariants of the elastic energy. We also discuss how to choose the numerical fluxes at inter-element boundaries and at physical boundaries. Section 4 briefly outlines the discretization and the implementation of the method. In Section 5 we present a comprehensive collection of numerical experiments illustrating the method's performance for manufactured solutions, for classical problems such as the propagation of Rayleigh, Lamb and Stoneley waves, as well as mode conversion at an interface between two solids. We conclude the numerical experiments with two applications: wave propagation in a slab with a stiff inclusion and an application in uncertainty quantification for non-destructive testing. The last section summarizes the paper and discusses possible extensions and improvements.

Finally, we again note that the computer codes used in the examples below are freely available from [19].

[1] As far as we know, the dG dispersion relation for elastic surface or interface waves has not been studied yet but we expect it to behave similarly as for other waves.