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A Galerkin isogeometric method for Karhunen–Loève approximation of random fields☆

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Abstract

This paper marks the debut of a Galerkin isogeometric method for solving a Fredholm integral eigenvalue problem, enabling random field discretization by means of the Karhunen–Loève expansion. The method involves a Galerkin projection onto a finite-dimensional subspace of a Hilbert space, basis splines (B-splines) and non-uniform rational B-splines (NURBS) spanning the subspace, and standard methods of eigensolutions. Compared with the existing Galerkin methods, such as the finite-element and mesh-free methods, the NURBS-based isogeometric method upholds exact geometrical representation of the physical or computational domain and exploits regularity of basis functions delivering globally smooth eigensolutions. Therefore, the introduction of the isogeometric method for random field discretization is not only new; it also offers a few computational advantages over existing methods. In the big picture, the use of NURBS for random field discretization enriches the isogeometric paradigm. As a result, an uncertainty quantification pipeline of the future can be envisioned where geometric modeling, stress analysis, and stochastic simulation are all integrated using the same building blocks of NURBS. Three numerical examples, including a three-dimensional random field discretization problem, illustrate the accuracy and convergence properties of the isogeometric method for obtaining eigensolutions.

Keywords: B-splines; NURBS; Fredholm integral eigenvalue problem; Hilbert–Schmidt operator; Uncertainty quantification

1. Introduction

Many uncertainty quantification problems in engineering and applied sciences require modeling spatial variability of random input parameters. For instance, the tensile and fracture toughness properties of engineering materials, the size and shape characteristics of mechanical components, and the wind and snow loads in structural systems all exhibit randomness that varies not only from sample to sample, but also from point to point in their respective domains. Therefore, random field treatment of spatial varying randomness is a vital ingredient in computational analysis. Loosely speaking, a random field represents a random quantity at each point of the domain and, therefore,

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engenders an infinite number of random variables. In practice, though, the number of random variables must be finite and manageable but also large enough to ensure an optimal or accurate approximation of the original random field. This process is often referred to as random field discretization.

A number of methods and approaches are available for random field discretization. For brevity, this paper will not perform a comprehensive review, but will direct readers to a paper by Betz et al. [1], which provides a summary of existing works, including many references cited therein. A popular approach, known by the name of Karhunen–Loève (KL) expansion [2–4], entails spectral decomposition of the covariance function, leading to an infinite series consisting of deterministic functions of space and uncorrelated random variables. The expansion is well known with diverse applications in engineering and applied sciences [5]. However, the KL expansion mandates solution of a Fredholm integral eigenvalue problem [6], which is not an easy task in general. Analytical solutions are available only when the covariance function has simpler functional forms, such as exponential or linear functions, and/or the problem domain is rectangular. For arbitrary covariance functions or arbitrary domains in two or three dimensions, numerical methods are often needed to solve the eigenvalue problem approximately.

For numerical solution of the integral eigenvalue problem, a well-known method is the Galerkin finite-element method (FEM) employed by Ghanem and Spanos [7] in the 1990s. Roughly speaking, the finite-element solution consists of a variational formulation and function spaces defined by its basis functions [8]. These basis functions are described by local representations via finite elements, resulting in a mesh or grid, which constitutes a non-overlapping decomposition of the computational domain into elementary shapes, such as triangles or tetrahedra and quadrilaterals or hexahedra. However, for mechanical systems with complex geometry, a finite-element mesh is often created from a computer-aided design (CAD) model, where the former is an approximation of the latter. Therefore, an additional source of imprecision is embedded in the FEM-based eigensolution. Another Galerkin approach, which sidesteps the need for element-wise decomposition, is the meshless or mesh-free method, especially the element-free Galerkin method [9], upon which Rahman and Xu [10,11] capitalize for the solution of the integral eigenvalue problem. The fundamental aspects of both FEM and the mesh-free method are identical as they are rooted in the same Galerkin formulation, but the function spaces and their basis functions are different: in FEM, the basis functions are interpolatory polynomials with C^0-continuity across element boundaries, whereas in the mesh-free method, the basis functions are non-interpolatory rational functions with at least C^1-continuity everywhere. In consequence, the approximate eigenfunctions of the KL expansion obtained by the mesh-free method are usually globally smoother than those derived from FEM. Nonetheless, as in FEM, the link between the mesh-free method and CAD geometry is, at best, tenuous [12]. Indeed, FEM or the mesh-free method may never faithfully replicate the CAD geometry. More importantly, for complex engineering designs, generating a high-quality finite-element mesh or mesh-free discretization from the CAD geometry is more formidable than performing the analysis. This is the principal motivation behind replacing finite-element- or mesh-free-generated basis functions with CAD-generated basis functions for solving the integral eigenvalue problem directly, leading to effective random field discretization.

This paper presents a Galerkin isogeometric method for solving the integral eigenvalue problem stemming from the KL expansion of a random field with an arbitrary covariance function and an arbitrary computational domain in three dimensions. The method entails performing a Galerkin discretization of the integral eigenvalue problem, formulation of the associated matrix eigenvalue problem by constructing the isogeometric function spaces spanned by basis splines (B-splines) and non-uniform rational B-splines (NURBS), and solution of the resultant matrix eigenvalue problem by standard methods. The paper is organized as follows. A brief exposition of NURBS paraphernalia and isogeometric concept is given in Section 2. Section 3 formally defines a random field and its KL expansion, followed by truncation of the KL expansion and a description of associated error measures. The limitation of the KL expansion is also discussed. Section 4 presents the proposed isogeometric method for solving the integral eigenvalue problem. The properties and construction of system matrices involved in the matrix eigenvalue problem are explained. The results from three numerical examples of increasing dimensions and hence complexity are reported in Section 5 and Appendix. Section 6 discusses future work. Finally, conclusions are drawn in Section 7.

2. Isogeometric analysis

Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_0^+ := [0, +\infty)$, and $\mathbb{R}^+ := (0, +\infty)$ represent the sets of positive integer (natural), non-negative integer, real, non-negative real, and positive real numbers, respectively. Denote by d the dimension of the physical or computational domain $\mathcal{D}$ of a geometrical object, which can be a curve, surface,