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Error estimates of B-spline based finite-element methods for the stationary quasi-geostrophic equations of the ocean

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Abstract

This paper presents theoretical error estimates of B-spline based finite-element methods for the streamfunction formulation of the stationary quasi-geostrophic equations, which describe the large scale wind-driven ocean circulation. We introduce variational formulations of the streamfunction formulation inspired by the interior penalty discontinuous Galerkin method. Dirichlet boundary conditions are weakly enforced in the formulations and stabilizations are achieved via Nitsche's method. Existence and uniqueness of the approximation are proved and optimal error estimates in the energy norm are demonstrated under a small data assumption. Numerical experiments are performed to verify the theoretical error estimates on rectangular and L-shape geometries.

Keywords: Nitsche's method; Ocean circulation; Streamfunction; Stabilization; Fourth-order partial-differential-equation; B-spline based finite-element

1. Introduction

The quasi-geostrophic equations (QGE) are one of the popular mathematical models used for the study of large-scale wind-driven ocean circulation. Physically, the QGE can capture the typical features of the oceanic flows such as the formation of strong western boundary currents, weak interior flows, and weak eastern boundary currents like those exhibited by the north Atlantic and Pacific oceans (Vallis [1] and Pedlosky [2]). From numerical point of view, the QGE allow for easy and efficient computational simulations of the ocean circulation based on two mathematical formulations, i.e., the mixed streamfunction vorticity formulation and the streamfunction formulation.

The mixed formulation is a system of second-order partial differential equations (PDE) requiring C^0-elements for standard conforming finite-element discretizations. On the other hand, the streamfunction formulation is fourth-order and thus, requires higher-order C^1-elements that make implementation more challenging. While available

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error estimates for the mixed formulation are suboptimal (Fix [3]), optimal convergence rates for the streamfunction formulation can be obtained as shown in Foster et al. [4]. For the streamfunction formulation of the stationary QGE (SQGE), Kim et al. [5] recently introduced a variational formulation based on the idea of Nitsche's [6] method for non-interpolatory basis functions such as B-splines. With cubic B-splines, the convergence study of the method was performed via numerical investigation without any theoretical error analysis. In this paper, we provide theoretical error estimates of the method along with their numerical verification using several examples.

Finite-element approximations using B-splines have two main distinct advantages. First, a high order continuity can be obtained at a relatively low computational cost for conforming finite-element discretizations of higher order PDEs. Second, curved and complex geometries can be represented exactly, as in isogeometric analysis (Hughes et al. [7]). Despite these advantages, B-splines are non-interpolatory that makes imposing even simple boundary conditions problematic. The imposition of boundary conditions in a strong manner may not be appropriate for problems where the formation of boundary layers is important. This is because doing so may induce artificial oscillations and may also result in decreasing the accuracy of a numerical method (Bazilevs and Hughes [8]). Kim and his colleagues [5,9,10] applied B-splines to the SQGE and two linear models to take advantage of the benefits of B-splines. They developed a variational formulation for the streamfunction formulation of the SQGE which is valid for non-interpolatory basis functions such as B-splines. A distinguishing feature of this formulation is to weakly enforce Dirichlet boundary conditions and its stabilization is achieved via Nitsche's method. As an application of this formulation, Jiang and Kim [11] modeled the ocean circulation problems with arbitrary shaped coastal lines on embedded interfaces.

Recently, Nitsche's method has been successfully applied to weakly impose boundary and interfacial conditions for the second- and fourth-order PDEs (Embar et al. [12] and Kim et al. [13]), the second-gradient theory (Kim and Dolbow [14] and Kim et al. [15–18]) and meshfree (Fernàndez-Mèndez and Huerta [19]) and embedded finite-element methods (Dolbow and Harari [20] and Hansbo and Hansbo [21]). The use of Nitsche's method to impose boundary conditions on complex geometries also generalizes easily to embedded finite-element methods, including the immersogeometric approach of Kamensky et al. [22] and the formulation of Nitsche's method due to Jiang et al. [23]. Kim et al. [24] applied the idea of Nitsche's method to the simulation of the streamfunction formulation of the SQGE using C^0-elements.

The main goal of this paper is two-fold. First, we introduce a slightly modified version of the Nitsche-type variational formulation introduced by Kim et al. [5]. The modified version facilitates the analysis of well-posedness of the discrete problem. We also consider its variants inspired by interior penalty discontinuous Galerkin (IPDG) method (Babǔska et al. [25] and Dawson et al. [26]). The optimal error estimate for all variants in the energy norm is derived, i.e., $\|\psi - \psi^h\|_h = \mathcal{O}(h^{k-1})$ where $k \geq 3$ is the polynomial degree for B-spline space. Second, numerical experiments are performed to verify the error estimate through several benchmark problems both on rectangular and L-shape geometries using cubic B-splines.

The remainder of the paper is organized as follows. In Section 2, we first introduce notations and useful lemmas with energy and continuity norms. In Section 3, we briefly describe the streamfunction formulation of the SQGE along with its B-spline based finite element formulation relevant to our consideration. Section 4 is devoted to the optimal error estimate for the Nitsche-type variational formulation of the SQGE. In Section 5, we demonstrate the analysis in Section 4 with two benchmark examples in rectangular and L-shape geometries. Finally, conclusions and a brief discussion of potential future research directions are contained in Section 6.

2. Preliminaries

In a one-dimensional B-spline approximation space, a function v_h can be written as

$$v_h(x) = \sum_{i=1}^{n} c_i B_i^k(x),$$

where c_i is the coefficient corresponding to the B-spline basis function $B_i^k(x)$ of degree k. In order to construct B-spline basis functions, we introduce a non-decreasing set of knots $\{p_i\}$, also known as knot vector p,

$$p = \{p_1 < p_2 < \cdots < p_{n+k+1}\},$$

where n is the number of basis functions. Then, the basis function can be generated by using the recurrence relation

$$B_i^k(x) = \left(\frac{x - p_i}{p_{i+k} - p_i}\right) B_i^{k-1}(x) + \left(\frac{p_{i+k+1} - x}{p_{i+k+1} - p_{i+1}}\right) B_{i+1}^{k-1}(x) \quad (k \geq 1),$$