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Local projection stabilization for the Stokes equation with Neumann condition

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Abstract

The local projection stabilization (LPS) method is already an established method for stabilizing saddle-point problems and convection–diffusion problems. The a priori error analysis is usually done for homogeneous Dirichlet data. It turns out that without Dirichlet conditions the situation is more involved, because additional boundary terms appear in the analysis. The standard approach can be modified by using additional stabilization terms on the non-Dirichlet boundary parts. We show that such terms lead to a similar a priori estimate as the classical LPS method but in a stronger norm.

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1. Introduction

The Stokes system is an important system of equations to model viscous incompressible flows. Due to its saddle-point structure the well-posedness can be obtained for appropriate function spaces by an inf–sup condition. If the discrete counterpart, e.g by finite elements, does not fulfill a corresponding discrete inf–sup condition, stabilization terms must be added to ensure existence and uniqueness for the pressure variable. This is for instance the case for equal-order finite elements, i.e. the same polynomial degree for the velocity and the pressure variables. There are well-known strategies for such stabilization techniques: Perhaps the most prominent ones are the pressure-stabilization/Petrov–Galerkin (PSPG) method [1], the local projection stabilization (LPS) [2,3]. and the interior penalty method [4]. Apart of saddle point problems, the LPS method has also been applied and analyzed in the context of convection stabilization, see e.g. the work of Barrenechea et al. [5,6] and Knobloch [7,8]. Related ideas are based on global pressure projections [9] or polynomial pressure projections, see [10].

The LPS method is usually formulated for the case of Dirichlet conditions on the entire boundary of the computational domain. Although other boundary conditions as outflow conditions (or Neumann conditions) are very important for many applications, the analysis for LPS does not yet cover this case. The reason is that the stability proof

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and the a priori estimate make use of integration by parts. For velocities with non-vanishing traces on the boundary of the domain, this partial integration generates boundary integrals which in general do not vanish and cannot be properly bounded by the standard LPS method. This work will extend the well-known results of LPS [3,11] for such kind of non-Dirichlet conditions. It turns out that the approach has to be modified by considering additional boundary fluctuations in the stabilizing term.

In Section 2 we formulate the underlying system of equations, the boundary conditions and its variational formulation. The non-standard inf–sup condition for the infinite dimensional function spaces and non-Dirichlet conditions is given in Section 3. In Section 4 we introduce the finite element spaces and the discrete counterpart of the Stokes system including the LPS stabilization. The analysis is carried out in Section 5; we present in particular the discrete inf–sup condition and an a priori error estimate on basis of the existence of an interpolation operator with certain orthogonality properties. We also show sufficient conditions to ensure the existence of such an interpolant, and we verify them for several finite elements. We end in Section 6 with numerical results.

2. Stokes system and its variational formulation

We consider the Stokes equations in a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, with a boundary split into two parts, $\partial\Omega = \Gamma_D \cup \Gamma_N$, homogeneous Dirichlet conditions on Γ_D and a natural outflow condition (zero stress) on Γ_N. With the velocity field $\mathbf{u} : \Omega \to \mathbb{R}^d$, the pressure $p : \Omega \to \mathbb{R}$, and a forcing term $\mathbf{f} : \Omega \to \mathbb{R}^d$ the Stokes system reads

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D, \tag{3}$$

$$\nabla\mathbf{u}\cdot\mathbf{n} - p\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_N. \tag{4}$$

The Hilbert spaces for $\mathbf{u}$, p are denoted by

$$\mathbf{V} := \{\mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\},$$
$$Q := L^2(\Omega).$$

We use the usual notations: $(\cdot,\cdot)_\omega$ for the L^2 scalar product in $\omega \subseteq \Omega$, $\|\cdot\|_\omega$ for the L^2-norm in ω. In the case $\omega = \Omega$, we suppress the index ω. For the semi-norm of the Sobolev space $H^l(\omega)$ we use also the standard notation $|\cdot|_{H^l(\omega)}$. By $\|\cdot\|_{\Gamma_N}$ we denote the $L^2(\Gamma_N)$-norm on Γ_N in the sense of the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(\Gamma_N)$. The corresponding bilinear form $A : (\mathbf{V}\times Q)\times(\mathbf{V}\times Q) \to \mathbb{R}$ for the Stokes equations is given by

$$A(\mathbf{u}, p; \boldsymbol{\phi}, \chi) := (\nabla\mathbf{u}, \nabla\boldsymbol{\phi}) - (p, \operatorname{div}\boldsymbol{\phi}) + (\operatorname{div}\mathbf{u}, \chi),$$

where $\boldsymbol{\phi} \in \mathbf{V}$ and $\chi \in Q$ are the test functions. The Stokes system in variational formulation reads

$$\mathbf{u} \in \mathbf{V},\ p \in Q: \quad A(\mathbf{u}, p; \boldsymbol{\phi}, \chi) = (\mathbf{f}, \phi) \qquad \forall\boldsymbol{\phi} \in \mathbf{V},\ \chi \in Q. \tag{5}$$

Note that $H_0^1(\Omega)^d \subseteq \mathbf{V}$ but $H_0^1(\Omega)^d \neq \mathbf{V}$ in the case of $\mathcal{H}^{d-1}(\Gamma_N) > 0$.

This variational formulation is consistent in the following sense: Each classical solution $(\mathbf{u}, p) \in (C^2(\Omega)^d \cap C(\bar{\Omega})^d) \times (C^1(\Omega) \cap C(\bar{\Omega}))$ of (1)–(4) is also a weak solution of (5). Vice-versa, each weak solution of (5) in $\mathbf{V}\times Q$ with enough regularity is also a classical solution of (1)–(4).

3. The inf–sup condition for partial Dirichlet conditions

It is well known that the inf–sup condition holds for the Stokes system with homogeneous Dirichlet conditions on the entire boundary $\partial\Omega$, i.e. there exists a $\gamma > 0$ s.t. (cf. e.g. Girault & Raviart [12], p. 81)

$$\sup_{u\in H_0^1(\Omega)^d\setminus\{0\}} \frac{(p, \operatorname{div}\mathbf{u})}{\|p\|\|\nabla\mathbf{u}\|} \geq \gamma \qquad \forall p \in L_0^2(\Omega), \tag{6}$$

where $L_0^2(\Omega)$ is the space of all $p \in L^2(\Omega)$ with vanishing mean, i.e.

$$\bar{p} := |\Omega|^{-1}\int_\Omega p\,dx \;=\; 0.$$