Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 333 (2018) 74–93
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Predictor–corrector p- and hp-versions of the finite element method for Poisson's equation in polygonal domains

B. Nkemzi*, S. Tanekou

Department of Mathematics, Faculty of Science, University of Buea, Cameroon

Received 4 October 2017; received in revised form 4 January 2018; accepted 17 January 2018
Available online 2 February 2018

Abstract

We consider boundary value problems for the Poisson equation on polygonal domains with general nonhomogeneous mixed boundary conditions and derive, on the one hand, explicit extraction formulas for the coefficients of the singularities. On the other hand, the formulas are used to construct efficient adaptations for the h-, p- and hp-versions of the finite element method for the numerical treatment. A priori error estimates show that the h-version of the finite element algorithm exhibits the same rate of convergence as it is known for problems with smooth solutions. However, the principal results of the present work are the robust exponential convergence results for the p- and hp-versions of the finite element method on quasiuniform meshes. In fact, it is shown that if the input data (source term and boundary data) are piecewise analytic, then with appropriate choices of conforming finite element subspaces V_N of dimension $N \in \mathbb{N}$, the p- and hp-versions of the finite element algorithms on quasiuniform meshes yield approximate solutions u_N to the exact solution u that satisfy the estimates $\|u - u_N\|_{H^1(\Omega)} \leq C_1 \mathrm{e}^{-b_1 N^{\frac{2}{3}}}$ and $\|u - u_N\|_{H^1(\Omega)} \leq C_2 e^{-b_2 N^{\frac{1}{3}}}$, respectively. Several numerical experiments are included to illustrate the practical effectiveness of the proposed algorithms. The results show that the theoretical error analyses are attained within the range of engineering accuracy.

MSC: 35A15; 35B65; 35J05; 65N30

Keywords: Poisson equation; Singularities; Finite element methods; A priori error estimates

1. Introduction

It is well known that solutions of elliptic boundary value problems on two-dimensional domains may entail singularities which are due either to corners or abrupt change in boundary conditions (cf. [1–4]) and that these singularities severely reduce the accuracy of standard numerical methods of solution, for example, the finite element method (cf. [5–7]). It is therefore not surprising that several methods for improving the accuracy of the numerical solution and computer codes for this class of problems already exist. In fact, one can put the various strategies into

* Corresponding author.
E-mail address: nkemzi.boniface@gmail.com (B. Nkemzi).

https://doi.org/10.1016/j.cma.2018.01.027
0045-7825/

two principal groups. Firstly, the geometric approach which consists of uniformly grading the finite element meshes towards the geometric singularities or relocating the nodes. These methods are most suitable for the h-version of the finite element method with polynomial order $p = 1$ and they have the advantage that the exact structure of the singular solutions is not required. It has been shown that with properly designed meshes these methods achieve the optimal rate of convergence (cf. [8–10]) and the references cited therein. A more superior technique under the geometric approach is the so-called geometric mesh refinement towards the singular points. It has been shown that by proper combination of mesh refinement and increasing polynomial order, the hp-finite element method can achieve exponential convergence for a large class of elliptic boundary value problems in one- and two-dimensional domains (cf. [11–15] and also [16–18]). The second strategy, is the functional approach. This includes the singular function method or Fix method which consists of augmenting the space of trial functions of the finite element method by lower order singular functions in the asymptotic expansion of the solution (cf. [7,19]). However, this method sometimes fails to approximate accurately the coefficients of the singularities (cf. [20]) and this leads to bad convergence of the numerical solution near the singular points. Another strategy under this category involves calculating a priori the coefficients in the asymptotic expansion of the solution (cf. [21–31]). In all these references and those cited therein, only the h-finite element method has been successfully implemented.

It is well known that when applicable the p- and hp-versions of the finite element method are far more superior over the h-version in terms of accuracy per the number of degrees of freedom, that is, the dimension of the resulting linear system (cf. [11,14,32]). For elliptic boundary value problems on two-dimensional domains with corner singularities, it is still worthwhile to consider and analyse such methods that would allow an efficient implementation of the p- and hp-versions of the finite element method on quasiuniform meshes and with high accuracy.

The main purpose of the present paper is threefold. Firstly, to propose extraction formulas for the computation of the coefficients of the singularities for boundary value problems for the Poisson equation on polygonal domains and with general nonhomogeneous mixed boundary conditions. Secondly, to present a new adaptive procedure for the h-, p- and hp-versions of the finite element method on quasiuniform meshes for the numerical solution. Finally, to demonstrate by means of numerical experiments the practical effectiveness of the new algorithm.

Our algorithm is based on the extraction formulas for the coefficients of the singularities and since it consists of computing an initial approximation of the solution using lower order finite element strategies on a coarse quasiuniform mesh, then correcting the approximation iteratively by means of higher order finite element strategies and smoothing the error, we call the new procedure a *predictor–corrector h-, p- or hp-finite element method*. The performance of the new algorithm is analysed by carrying out various error estimates and it is shown that the rate of convergence of the h-version of the algorithm on quasiuniform meshes is the same as it is known for problems with smooth solutions in the sense of the shift theorem. However, the principal results of the present work are the robust exponential convergence results for the p- and hp-versions of the finite element algorithm. In fact, it is shown that if the input data (source term and boundary data) are piecewise analytic, then with appropriate choices of conforming finite element subspaces V_N of dimension $N \in \mathbb{N}$, the p- and hp-versions of the finite element algorithms on quasiuniform meshes yield approximate solutions u_N to the exact solution u that satisfy the estimates

$$\|u - u_N\|_{H^1(\Omega)} \leq C_1 \mathrm{e}^{-b_1 N^{\frac{2}{3}}} \quad \text{and} \quad \|u - u_N\|_{H^1(\Omega)} \leq C_2 \mathrm{e}^{-b_2 N^{\frac{1}{3}}}, \tag{1.1}$$

respectively, where the C_i and the b_i are positive real constants. The present algorithm, as we shall see in Section 4, is different from the existing algorithms in two principal aspects. Firstly, the finite element spaces are not augmented with singular functions as it is the case with the singular function method. Rather the singular part of the solution is computed separately from the regular part. Secondly, the extraction procedure for the coefficients of the singularities is different and more efficient as is the case, for example, in [22].

It has been known that the p-finite element method exhibits exponential convergence if the solution of the boundary value problem is analytic up to the boundary. However, this result has never been achieved practically. We observe from the inequalities in (1.1) that the p-algorithm is far more superior than the hp-method. In the practical implementation of the algorithms, the resulting linear systems are sparse and only the load vectors are modified during the corrector process. Thus the great advantage of the present method is that existing finite element softwares can be adapted in a relatively easy manner.

This paper is organized as follows: In Section 2, we introduce the boundary value problem and collect some facts about the asymptotic properties of the solution. Section 3 contains the derivation of extraction formulas for the coefficients of the various forms of singularities. Section 4 contains a brief description of the usual finite element