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# A random field model for anisotropic strain energy functions and its application for uncertainty quantification in vascular mechanics

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**Abstract**

This paper deals with the construction of random field models for spatially-dependent anisotropic strain energy functions indexed by complex geometries. The approach relies on information theory and the principle of maximum entropy, which are invoked in order to construct the family of first-order marginal probability distributions in accordance with fundamental constraints such as polyconvexity, coerciveness and consistency at small strains. We then address the definition of a sampling methodology able to perform on domains that are non-homotopic to a sphere, with the aim to generate the non-Gaussian random fields on non-simplified geometries—such as patient-specific geometries in computational biomechanics. The algorithm is based on the construction of a diffusion field that involves local geometrical features of the manifolds defining domain boundaries. We finally present numerical applications on vascular tissues, including the case of an arterial wall defined by real patient-specific data.


*Keywords:* Stochastic modeling; Hyperelasticity; Nonlinear elasticity; Soft biological tissues; Entropy; Uncertainty quantification

## 1. Introduction

The mathematical representation of uncertainties is a key ingredient in predictive science and engineering [1]. In continuum mechanics, uncertainties in constitutive laws can arise from subscale (morphological) randomness, especially when the so-called separation of scales does not hold [2], or from fluctuations in the properties of the constitutive phases. This variability can notably be inherited from processing in the case of engineered composites, or from various factors including age, gender or physiological state when biological tissues are concerned. Experimental evidences of stochasticity in the response of biological tissues can be found in e.g. [3–14]. In this context, the question as to how properly model, identify and simulate such randomness is a central challenge in both computational mechanics and mechanics of materials.

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Within the framework of uncertainty quantification [15,16], popularized by the seminal work from Ghanem and Spanos [17], this stochastic modeling task has been mostly addressed by resorting to polynomial chaos expansions (PCE) [18,19] of random variables [20,21] and random fields [22]. In particular, detailed derivations focusing on the PCE-based representation of tensor-valued coefficients for stochastic elliptic operators (hence encompassing the case of linear elasticity) can be found in the recent work [23]. Statistical approaches based on Bayesian inference have also received considerable attention, with the promise to accommodate data paucity through data-driven methodologies [1] (see also [24] for an application to PCE). Algebraic decompositions of random fields for stochastic three-dimensional elasticity were further constructed in [25–30] and validated on various multiscale or multimodel problems in e.g. [31–33]. Spectral expansions for elasticity random fields were finally obtained in [34] for all material symmetry classes.

In contrast, the construction of stochastic models in nonlinear elasticity has received much less consideration to date, and most of the efforts were basically focused on the propagation of PCE through nonlinear computational models (see e.g. [35,36] for applications in elastoplasticity). A noticeable contribution relying on a Bayesian formulation and accounting for both spatial variability and non-simplified domains can be found in [37]. Here, a parameter defining the isotropic strain energy function of interest is modeled as a non-Gaussian (translation) random field for which (i) the first-order marginal probability function is selected a priori; (ii) the covariance (exponential) kernel defining the underlying centered Gaussian field is chosen with no reliance on the geometry (see also [38]). The construction of information-theoretic stochastic models for random isotropic strain energy functions was recently tackled in [39–41] (see the references therein for surveys), for both compressible and incompressible materials. These models, which can appropriately be used in order to define first-order marginal probability laws for random fields, were further identified and validated with experiments on various soft biological tissues (including brain and liver tissues, as well as spinal cord white matter) in [41].

The aim of this paper is to address the construction of a prior stochastic model for spatially-dependent anisotropic strain energy functions. Such a prior model essentially allows *admissible* samples of the strain energy functions to be drawn, in accordance with fundamental theoretical results in finite elasticity, and can subsequently be used to solve underdetermined statistical inverse problems or combined with Bayesian approaches [42]. Envisioned applications include large-scale computational analysis in biomechanics, particularly for soft biological tissues such as vascular vessels, and the modeling of composite materials at large strains. This construction raises two main challenging issues. First of all, the consideration of anisotropic strain energy functions leads to higher-dimensional parametrizations and more complex constraints between the variables. In addition, it requires the construction of stochastic models for non-Gaussian vector-valued random fields. Second, the simulation of these models necessitates describing the correlation structure and sampling the random fields on non-simplified geometries where boundaries typically take the form of smooth manifolds.

The rest of this paper is organized as follows. The fundamentals of continuum mechanics and hyperelasticity are first briefly exposed in Section 2. In particular, the definition of a prototypical strain energy function is presented, and theoretical requirements raised by the analysis of the nonlinear boundary value problems are highlighted. The stochastic framework is next introduced in Section 3. Here, a new random field model is derived within the framework of information theory. This probabilistic modeling effort is subsequently complemented, in Section 4, with the definition of a sampling methodology. The computational approach relies on solving a stochastic partial differential equation, the coefficients of which are specifically defined in accordance with the geometry under consideration. This allows, in particular, the case of non-simplified domains to be handled in an efficient and robust manner. Various applications, including the case of an arterial wall defined by a patient-specific geometry and undergoing inner pressure, are finally presented in Section 5 to assess and illustrate the modeling capability of the proposed stochastic framework and algorithms.

## 2. Deterministic modeling of anisotropic hyperelastic materials

### 2.1. Kinematics

In this section, we recall the continuum mechanics framework that is relevant to the modeling of anisotropic materials at finite strains. The interested reader is referred to classical textbooks (e.g. [43–45]) for a more extensive exposure.