Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 333 (2018) 147–175
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Time domain boundary elements for dynamic contact problems

Heiko Gimperlein[a,b,*], Fabian Meyer[c], Ceyhun Özdemir[d], Ernst P. Stephan[d]

[a] *Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot-Watt University, Edinburgh, EH144 AS, United Kingdom*
[b] *Institute for Mathematics, University of Paderborn, Warburger Str. 100, 33098 Paderborn, Germany*
[c] *Institute of Applied Analysis and Numerical Simulation, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany*
[d] *Institute of Applied Mathematics, Leibniz University Hannover, 30167 Hannover, Germany*

Received 22 March 2017; received in revised form 7 January 2018; accepted 17 January 2018
Available online 31 January 2018

Dedicated to Erwin Stein on the occasion of his 85th birthday

Abstract

This article considers a unilateral contact problem for the wave equation. The problem is reduced to a variational inequality for the Dirichlet-to-Neumann operator for the wave equation on the boundary, which is solved in a saddle point formulation using boundary elements in the time domain. As a model problem, also a variational inequality for the single layer operator is considered. A priori estimates are obtained for Galerkin approximations both to the variational inequality and the mixed formulation in the case of a flat contact area, where the existence of solutions to the continuous problem is known. Numerical experiments demonstrate the performance of the proposed mixed method. They indicate the stability and convergence beyond flat geometries.

Keywords: Boundary element method; Variational inequality; Mixed method; A priori error estimates; Wave equation

1. Introduction

Contact problems play an important role in numerous applications in mechanics, from fracture dynamics and crash tests to rolling car tires [1]. As the contact takes place at the interface of two materials, for time-independent problems boundary elements and coupled finite/boundary elements provide an efficient and much-studied tool for numerical simulations [2,3]. The analysis of such problems is well-understood in the context of elliptic variational inequalities.

While contact for time-dependent problems is of clear practical relevance, neither its analysis nor rigorous boundary element methods have been much explored. There is an extensive computational literature, including [4–9], but analytically even the existence of solutions to these free boundary problems is only known for flat contact area [10,11]. Some rigorous results have recently been obtained for Nitsche stabilized finite elements [12].

* Corresponding author at: Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot-Watt University, Edinburgh, EH144 AS, United Kingdom.
E-mail address: h.gimperlein@hw.ac.uk (H. Gimperlein).

https://doi.org/10.1016/j.cma.2018.01.025
0045-7825/

In this work we propose a time domain boundary element method for a $3d$ dynamic contact problem in the case of the scalar wave equation, as a model problem for elasticity. We provide a priori error estimates for our numerical scheme in the case of a flat contact area, and our numerical experiments indicate the convergence and efficiency also for curved contact geometries. Motivating references from the time-independent setting include [13,14].

For the precise statement of the problem, consider a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$, where either Ω is bounded or $\Omega = \mathbb{R}^{n-1}$. Let G be a bounded Lipschitz subset of Γ. We consider a unilateral contact problem for the wave equation for the displacement $w : \mathbb{R} \times \Omega \to \mathbb{R}$. It corresponds to a simplified model for a crack in G between Ω and a non-penetrable material in $\mathbb{R}^n \setminus \overline{\Omega}$. The contact conditions for non-penetration are described in terms of the traction $-\mu \frac{\partial w}{\partial \nu}\big|_G$ and prescribed forces h:

$$\begin{cases} w|_{\mathbb{R}\times G} \geq 0, \ -\mu \dfrac{\partial w}{\partial \nu}\Big|_{\mathbb{R}\times G} \geq h, \\ w|_{\mathbb{R}\times G} > 0 \implies -\mu \dfrac{\partial w}{\partial \nu}\Big|_{\mathbb{R}\times G} = h. \end{cases} \tag{1}$$

The full system of equations for the contact problem is given by:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} = c_s^2 \Delta w, & \text{for } (t,x) \in \mathbb{R}\times\Omega, \\ w = 0, & \text{on } \mathbb{R}\times\Gamma\setminus G, \\ w \geq 0, \ -\mu\dfrac{\partial w}{\partial \nu} \geq h, & \text{on } \mathbb{R}\times G, \\ \left(-\mu \dfrac{\partial w}{\partial \nu} - h\right) w = 0, & \text{on } \mathbb{R}\times G, \\ w = 0, & \text{for } (t,x) \in (-\infty,0)\times\Omega \end{cases} \tag{2}$$

where c_s denotes the speed of the wave. While we focus on the physically relevant three dimensional case, $n = 3$, the analysis of the numerical schemes can be adapted to $n = 2$. Like for time-independent contact, a formulation as a nonlinear problem on the contact area G leads to efficient numerical approximations.

In this article we formulate (2) as a variational inequality on G in terms of the Dirichlet-to-Neumann operator (12) for the wave equation. Similar to time-independent problems, the Dirichlet-to-Neumann operator is computed in terms of boundary integral operators as $\frac{1}{2}(W - (1-K')V^{-1}(1-K))$, where V, K, K' and W are the layer potentials defined in (7)–(10). Because the contact area and contact forces are often relevant in applications, we replace the variational inequality for the Dirichlet-to-Neumann operator by an equivalent mixed system, which we discretize with a time domain Galerkin boundary element method. The resulting discretized nonlinear inequality in space–time simultaneously approximates the displacement w and the contact forces $-\mu\frac{\partial w}{\partial \nu}$ on G. It is solved with a Uzawa algorithm, either as a time-stepping scheme or in space–time.

The resulting boundary element method is analyzed in the case of a flat contact area, a situation where the existence of solutions to the contact problem (2) is known. We obtain a priori estimates for the numerical error, both for the variational inequality and a mixed formulation:

Theorem 1. *Let $h \in H_\sigma^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ and let $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$, respectively $u_{\Delta t,h} \in \tilde{K}^+_{t,h} \subset H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$ be the solutions of the continuous variational inequality (21), respectively its discretization (22). Then the following estimate holds:*

$$\|u - u_{\Delta t,h}\|^2_{-\frac{1}{2},\frac{1}{2},\sigma,\star} \lesssim_\sigma \inf_{\phi_{\Delta t,h}\in\tilde{K}^+_{t,h}} (\|h - p_Q \mathcal{S}_\sigma u\|_{\frac{1}{2},-\frac{1}{2},\sigma} \|u - \phi_{\Delta t,h}\|_{-\frac{1}{2},\frac{1}{2},\sigma,\star} + \|u - \phi_{\Delta t,h}\|^2_{\frac{1}{2},\frac{1}{2},\sigma,\star}). \tag{3}$$

This result is shown as Theorem 10 in Section 5.

Theorem 2. *The discrete mixed formulation (31) of the contact problem admits a unique solution. The following a priori estimates hold:*

$$\|\lambda - \lambda_{\Delta t_2,h_2}\|_{0,-\frac{1}{2},\sigma} \lesssim \inf_{\tilde{\lambda}_{\Delta t_2,h_2}} \|\lambda - \tilde{\lambda}_{\Delta t_2,h_2}\|_{0,-\frac{1}{2},\sigma} + (\Delta t_1)^{-\frac{1}{2}} \|u - u_{\Delta t_1,h_1}\|_{-\frac{1}{2},\frac{1}{2},\sigma,*}, \tag{4}$$