Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 333 (2018) 287–310
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations

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Received 11 August 2017; received in revised form 27 December 2017; accepted 17 January 2018

Abstract

In this paper we present and analyze a hybridizable discontinuous Galerkin (HDG) method for a mixed curl–curl formulation of the steady state coercive Maxwell equations. With a discrete Sobolev embedding type estimates for the discontinuous polynomials, we provide a priori error estimates for the electric field and the Lagrange multiplier in the energy norm. With the smooth or minimal regularity assumption on the exact solution, we have optimal convergence rate for the electric field and the Lagrange multiplier in the energy norm. The a priori error estimate for the electric field in the L^2-norm is also obtained by the duality argument, and the approximation is also optimal for the electric field in the L^2-norm. Moreover, by employing suitable Helmholtz decompositions of the error, together with the upper bound estimate for the Lagrange multiplier, we provide a computable residual-based a posteriori error estimator which is derived based on the error measured in terms of a mesh-dependent energy norm. The efficiency of the a posteriori error estimator is also established. Three dimensional numerical results testing the performance of the a priori and a posteriori error estimates for the Maxwell equations are given.

MSC: 65N15; 65N30

Keywords: Discontinuous Galerkin; Hybridization; Maxwell equations; A priori error estimate; A posteriori error estimate

1. Introduction

In this paper, we consider the steady state coercive Maxwell equations as follows:

$$\nabla \times (\nabla \times \boldsymbol{u}) + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

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https://doi.org/10.1016/j.cma.2018.01.030

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \partial\Omega, \tag{1.1c}$$
$$p = 0 \qquad \text{on } \partial\Omega, \tag{1.1d}$$

where Ω is a simply connected polyhedral Lipschitz domain in $\mathbb{R}^3$ and the unknowns are electric field $\boldsymbol{u}$ and pseudo-pressure p. Here $\boldsymbol{f} \in H(\text{div}, \Omega)$ is the source term, $\boldsymbol{g}$ is given on $\partial\Omega$ and $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$.

In the literature, various numerical methods have been investigated for the Maxwell equations. For instance, the $H(\text{curl}, \Omega)$-conforming edge element methods [1–5], discontinuous Galerkin (DG) method [6–17], and discontinuous Petrov–Galerkin (DPG) method [18]. The DG methods have several attractive features which include their robustness, element-wise conservation properties, some flexible capabilities in mesh generation and to provide high-order accurate solutions. Especially, comparing with $H(\text{curl}, \Omega)$-conforming edge element methods, it is much easier to implement DG methods using hp-adaptivity for the Maxwell equations. However, the total number of global degrees of freedom of DG methods is much more than that of $H(\text{curl}, \Omega)$-conforming edge element methods. In fact, after static condensation, the global unknowns of DG methods are the traces of numerical electric field and numerical pseudo-pressure from both sides of each interior interface on the mesh skeleton, while those of $H(\text{curl}, \Omega)$-conforming edge element methods are the tangential trace of numerical electric field and trace of numerical pseudo-pressure on each interior interface on the mesh skeleton. So, if we assume that all methods use the same three dimensional simplicial mesh with the same polynomial order, the ratio between the global unknowns of DG methods and that of $H(\text{curl}, \Omega)$-conforming edge element methods is about $\frac{8}{3}$. In order to reduce the dimension of system from the standard approximation DG space, the hybridizable discontinuous Galerkin (HDG) methods [19–24] were recently introduced. The resulting system of HDG methods is only due to the unknowns on the skeleton of the mesh. Two HDG methods were introduced (without error analysis) in [12] for the numerical solution of the time-harmonic Maxwell equations. Some coercive bilinear formulations (cf. e.g. [9,10,17] and the references therein) have been developed to solve the coercive Maxwell equations (1.1) in the curl–grad system. In a recent work [17], we considered two new HDG methods on simplicial mesh and general polyhedral mesh respectively for the above coercive Maxwell equations with smooth regularity assumption on the exact solution. Due to a careful design of the numerical flux and using a non-trivial subspace of polynomials to approximate the numerical tangential trace of the electric field, the HDG methods in [17] provide optimally convergent approximations to the electric field and achieves superconvergence for the electric field without postprocessing from the point of view of global degrees of freedom. However, the solution of the Maxwell equations usually has limited regularity due to the complexity of the domain. In this paper, we further consider an HDG method for the above coercive Maxwell equations with smooth or minimal regularity assumption on the exact solution. Compared with the HDG methods in [12,17], two stabilization parameters in the numerical fluxes are carefully designed to obtain the optimally convergent HDG method for the Maxwell equations (1.1). We would like to point out that the global unknowns of HDG method in this paper are $(\widehat{\boldsymbol{u}}_h^t, \widehat{p}_h)$ - the numerical approximations to the tangential trace of electric field and the trace of pseudo-pressure on the mesh skeleton. So, on the same simplicial mesh with the same polynomial order, the number of global degrees of freedom of HDG method is approximately the same as that of $H(\text{curl}, \Omega)$-conforming edge element methods.

In order to present the HDG method, we write the Maxwell equations (1.1) into a system of first order equations. We introduce a new unknown $\boldsymbol{w} = \nabla \times \boldsymbol{u}$. Now we can write the Maxwell equations (1.1) in a mixed curl–curl formulation as follows:

$$\nabla \times \boldsymbol{u} = \boldsymbol{w} \qquad \text{in } \Omega, \tag{1.2a}$$
$$\nabla \times \boldsymbol{w} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.2b}$$
$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.2c}$$
$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \partial\Omega, \tag{1.2d}$$
$$p = 0 \qquad \text{on } \partial\Omega. \tag{1.2e}$$

The objective of this paper is to develop an absolutely well-posed HDG method which deserves optimal approximation for the above mixed curl–curl formulation (1.2) with smooth or minimal regularity assumption on the exact solution. We use polynomials of degree $k, k, k+1$ in the HDG method to approximate $\boldsymbol{w}, \boldsymbol{u}$ and p respectively. We discovered a key discrete Sobolev embedding inequality: let $\boldsymbol{v} \in \boldsymbol{P}_k(\mathcal{T}_h)$ satisfying $(\boldsymbol{v}, \nabla q)_\Omega = 0$ for any $q \in H_0^1(\Omega) \cap P_{k+1}(\mathcal{T}_h)$, then

$$\|\boldsymbol{v}\|_{\mathcal{T}_h} \leq C\left(\|h^{-\frac{1}{2}}[\![\boldsymbol{v} \times \boldsymbol{n}]\!]\|_{\mathcal{E}_h} + \|\nabla \times \boldsymbol{v}\|_{\mathcal{T}_h}\right), \tag{1.3}$$