Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 333 (2018) 331–355
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A linear domain decomposition method for partially saturated flow in porous media

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Received 8 August 2017; received in revised form 22 December 2017; accepted 17 January 2018
Available online 2 February 2018

Abstract

The Richards equation is a nonlinear parabolic equation that is commonly used for modelling saturated/unsaturated flow in porous media. We assume that the medium occupies a bounded Lipschitz domain partitioned into two disjoint subdomains separated by a fixed interface Γ. This leads to two problems defined on the subdomains which are coupled through conditions expressing flux and pressure continuity at Γ. After an Euler implicit discretisation of the resulting nonlinear subproblems, a linear iterative (L-type) domain decomposition scheme is proposed. The convergence of the scheme is proved rigorously. In the last part we present numerical results that are in line with the theoretical finding, in particular the convergence of the scheme under mild restrictions on the time step size. We further compare the scheme to other approaches not making use of a domain decomposition. Namely, we compare to a Newton and a Picard scheme. We show that the proposed scheme is more stable than the Newton scheme while remaining comparable in computational time, even if no parallelisation is being adopted. After presenting a parametric study that can be used to optimise the proposed scheme, we briefly discuss the effect of parallelisation and give an example of a four-domain implementation.

Keywords: Domain decomposition; L-scheme linearisation; Richards equation

1. Introduction

Unsaturated flow processes through porous media appear in a variety of physical situations and applications. Notable examples are soil remediation, enhanced oil recovery, CO_2 storage, harvesting of geothermal energy, or the design of filters and fuel cells. Mathematical modelling and numerical simulation are essential for understanding

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https://doi.org/10.1016/j.cma.2018.01.029
0045-7825/

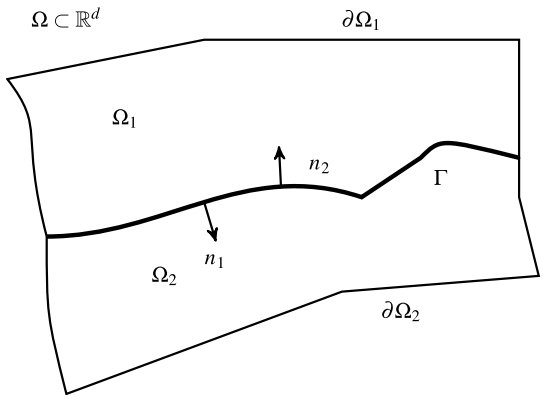

Fig. 1. Illustration of a layered soil domain $\Omega = \Omega_1 \cup \Omega_2 \subset \mathbb{R}^d$ with fixed interface Γ. Also shown are the normal vectors along the interface.

such processes, since measurements and experiments are very difficult if not impossible, and hence only limitedly available. The associated mathematical and computational challenges are manifold. The mathematical models are usually coupled systems of nonlinear partial differential equations and ordinary ones, involving largely varying physical properties and parameters, like porosity, permeability or soil composition. Together with the large scale and possible complexity of the domain, this poses significant computational challenges, making the design and analysis of robust discretisation methods a non-trivial task.

In this work we focus on saturated/unsaturated flow of one fluid (water) in a porous medium (e.g. the subsurface) occupying the domain $\Omega \subset \mathbb{R}^d$ ($d \in \{1, 2, 3\}$). Besides water, a second phase (air) is present, which is assumed to be at a constant (atmospheric) pressure. This situation is described by the Richards equation, here in pressure formulation

$$\Phi \partial_t S(p) - \nabla \cdot \left[\frac{\boldsymbol{K}}{\mu} k_r\big(S(p)\big) \nabla \big(p + z\big) \right] = 0, \tag{1}$$

see e.g. [1], originally [2,3]. In the above Φ denotes the porosity, S is the water saturation, p is the water pressure, k_r is the relative permeability, $\boldsymbol{K}$ the intrinsic permeability and $z = -\rho_w g x_3$ is the gravitational term in direction of the x_3-axis. Finally, g is the gravitational acceleration, ρ_w the water density and μ its viscosity. With $T > 0$ being a maximal time, the equation is defined for the time $t \in (0, T)$ on the bounded Lipschitz domain Ω.

Below we propose a domain decomposition (DD) scheme for the numerical solution of (1). To this aim we assume that Ω is partitioned into two subdomains Ω_l ($l \in \{1, 2\}$) separated by a Lipschitz-continuous interface Γ, see Fig. 1. In other words one has $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$. The restriction to two subdomains is made for the ease of presentation, but the scheme can be extended straightforwardly to more subdomains, see Remark 3 and Section 4.4. In each Ω_l ($l \in \{1, 2\}$) we use the physical pressure p_l as primary variable. Furthermore, the permeability and porosity in each of the subdomains may be different and even discontinuous, which is the case of a heterogeneous medium consisting of block-type heterogeneity (like a fractured medium).

In view of its relevance for manifold applications in real life, Richards' equation has been studied extensively, both analytically and numerically, and the dedicated literature is extremely rich. We restrict ourselves here by mentioning [4,5] for the existence of weak solutions and [6] for the uniqueness. Numerical schemes for the Richards equation, or in general for degenerate parabolic equations, are analysed in [7–15]. Most of the papers are considering the backward Euler method for the time discretisation in view of the low regularity of the solution, see [4], and to avoid restrictions on the time step size.

Different approaches with regard to spatial discretisation have been considered. Galerkin finite elements were used in [8,16,17]. Discontinuous Galerkin finite element schemes for flows through (heterogeneous) porous media have been studied in [18,19]. Finite volume schemes including multipoint flux approximation ones for the Richards equation are analysed in [20,21,13], and mixed finite elements in [7,22,10–12,15,14]. Such schemes are locally mass conservative.

Applying the Kirchhoff transformation [4] brings the mathematical model to a form that simplifies mathematical and numerical analysis, see e.g. [8,7,10,11]. However, the transformed unknown is not directly related to a physical