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Generalised path-following for well-behaved nonlinear structures

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Abstract

Recent years have seen a research revival in structural stability analysis. This renewed interest stems from a paradigm shift regarding the role of buckling instabilities in engineering design—from detrimental sources of catastrophic failure to novel opportunities for functionality. Novel nonlinear structures take the form of optimised thin-walled structures that operate safely in the post-buckling regime; shape-morphing structures that exploit multi-stability to snap and pop between different configurations; and meta-materials that derive novel material properties from a cascade of choreographed instabilities. Hence, elastic instabilities are no longer considered as structural failures but rather exploited for repeatable *well-behaved* adaptations. In this article we focus on shape-morphing—a bio-inspired design strategy that intends to conform structures to different operating conditions. Computational tools that integrate easily with established methods used in industry, and that are capable of capturing the full phase diagram of compound instabilities and entangled post-buckling paths typical of these structures, are limited. Such a capability is crucial, however, as confidence in predictive tools can be key in enabling non-conventional designs. One potential candidate in this regard is generalised path-following, which combines the computational robustness of numerical continuation algorithms with the geometric versatility of the finite element method. In this paper we collate an array of successful computational tools introduced by other researchers, and introduce our own developments, to present a modelling framework fit for analysing and designing with well-behaved nonlinear structures in industry and academia. Particularly, we show that the full complexity of multi-snap events of morphing composite laminates is robustly captured by generalised path-following algorithms, and that the ability to determine loci of singular points with respect to a set of parameters is especially useful for tracing the boundaries of bistability in parameter space. Furthermore, we shed new insight into the mechanics of multi-stable laminates, showing that the multi-stability and snapping behaviour of these structures is much richer than previously assumed, featuring many unstable post-buckling branches and localised regions of stability.

Keywords: Generalised path-following; Nonlinear structures; Multi-functionality; Bifurcations; Morphing; Composites

1. Introduction

To deliver the next generation of lightweight engineering structures, researchers and engineers are hoping to exploit, rather than avoid, elastic instabilities. Such designs could take the form of optimised thin-walled structures that operate

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safely in the post-buckling regime [1,2]; shape-morphing structures that snap between different configurations [3–11]; advanced meta-materials with innovative properties constructed from multi-scale arrangements of mechanically multi-stable components [12–18]; and other diverse applications such as self-encapsulating structures [19] and fluidic soft actuators [20].

Indeed, there is an ongoing trend of shifting our intuitions about structural instabilities as sources of catastrophic failure to opportunities for functionality [21]. It is well known that efficient and lightweight structures are prone to structural instabilities and collapse [22,23]. In this case, the predominant design philosophy is to prevent buckling or at least make its effects benign. On the other hand, when reconfigurations in shape or large elastic displacements are required, buckling is often encouraged [24]. Reis [21] recently reviewed the burgeoning research effort focusing on exploiting instabilities to enable novel designs, and therefore provided a new perspective on buckling, namely from *buckliphobia* to *buckliphilia*.

In the aerospace industry, shape-morphing structures are viewed as a promising technology to enable more structurally efficient designs [3–11]. The premise behind this idea is simple—if structures can be designed to adapt their shape to more optimally conform to different loading conditions, then structural efficiency is improved as a result. In fact, this multi-functionality has a very strong empirical proponent: nature. Birds, for example, can adapt the camber and angle of attack of their wings to different flight scenarios. Even though some of the concepts found in nature are already being exploited in aircraft structures, such as slats and flaps, they often rely on rigid load-bearing components connected to heavy hydraulic or electric actuators. This is where multi-stable structures are particularly attractive. By applying a suitable force, a multi-stable structure can be snapped from one stable state to another, thereby considerably reconfiguring its shape. Because each stable state is self-equilibrated, it does not require external energy to hold its shape, and additionally, the sensing, actuation and control functions are embedded within the nonlinear mechanics of the structure (passive control) without adding additional mass via ancillary devices.

The uptake of these novel designs in industry is partly hampered by a lack of robust computational tools tailored to the design of structures whose characteristic feature is a form of spatial chaos [25], *i.e.* equilibrium manifolds featuring an entire series of bifurcations that give rise to many equilibrium branches and possible loading histories, especially in cases where dynamic snap-buckling is exploited for shape adaptation. Predicting these features reliably is pushing well-established finite element techniques to their limit, and is creating an acute requirement for new computational approaches for analysis and design [21,26]. In recent years the focus has been on analytical and computational techniques constrained to the analysis of very specific morphing problems with particular load cases and geometries [27–36]. A drawback of these tailored approaches is that simplifying assumptions are often made, which prevent applicability to a wider range of problems. Examples include restrictions on the geometric nonlinearity to *von Kármán* strains (small strains, small displacements and moderate rotations); posing the problem on a domain that is cumbersome to extend beyond simple geometries; and using nonlinear stability analyses without robust branch-switching.

Due to its geometric versatility and developmental maturity, the finite element method is the preferred technique for modelling complex structural problems. Commercial finite element packages may be used to analyse multi-stable structures [37–39], but most of the time, these analyses are rather *ad hoc*, because the full taxonomy of stable and unstable equilibria cannot be revealed robustly using the quasi-static implicit solvers implemented in these codes. Rather, the engineer needs to be aware of possible bifurcation points *a priori*, and then "coax" the algorithm to land on a specific post-buckling mode shape using initial imperfections. Such an approach is cumbersome and requires user intervention, such that it becomes difficult and inefficient to explore the entire design nonlinear space.

To illustrate this point, an example is shown in Fig. 1, which describes a typical snap-through load–displacement path. The behaviour is complicated by the presence of a secondary equilibrium path branching from the unstable region of the equilibrium curve. In a physical experiment under rigid loading, this hypothetical structure would snap to point (c) upon reaching limit load (a). A path-following solver without the ability to pinpoint singular points and then branch-switch when needed, would generally pass the bifurcation (b) and continue to traverse along the unstable fundamental path. Not being aware of the existence of point (c), an engineer interpreting the results would then suggest that the structure snaps from point (a) to point (d), rather than from point (a) to point (c). To path-follow along the secondary equilibrium path by means of the often-used imperfection method, the analyst needs to be aware of the existence of bifurcation point (b), stop the path-following algorithm before reaching it, run a linear eigenvalue analysis and then apply the lowest eigenmode as an imperfection. Such a procedure is cumbersome to implement and restricted to simple bifurcation points (for compound bifurcations there is little control over which branch the solver