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ScienceDirect
Comput. Methods Appl. Mech. Engrg. 326 (2017) 338–357
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

# Eigenvector sensitivity when tracking modes with repeated eigenvalues

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Received 24 March 2017; received in revised form 2 June 2017; accepted 26 July 2017
Available online 12 August 2017

**Abstract**

While eigenvalue optimization problems have been widely studied during the last three decades, particularly for structural and topology optimization problems, there are very few examples of problems involving mode shapes either in the cost or the constraints. In this paper, we propose a general framework for computing eigenvector sensitivity whenever tracking specific mode shapes selected beforehand. Of course, the approach is valid for either non-repeated or repeated eigenvalues, but here the emphasis is placed on the multiple eigenvalues case. Both mathematical validity of the algorithm and numerical corroboration through several examples are included.


*Keywords:* Eigenvector derivative; Repeated eigenvalues; Mode shape derivative; Eigenmode optimization; Eigenmode selection; Modal assurance criterion

## 1. Introduction

Eigenvalue optimization problems appear very frequently when dealing with structural stability and vibration analysis, and nowadays it is a very well understood topic in topology optimization [1]. Eigenvalue sensitivity analysis for this kind of problems is a classic issue dating back to the sixties. For eigenvalue sensitivity calculation there are two different cases: simple, non-repeated, or multiple, repeated, eigenvalues, being this last case much more difficult and subtle than the former one, since multiple eigenvalues are not differentiable. There are many references where this has been addressed, and among those we cite [2,3]. Sensitivity analysis has been applied for optimization of eigenvalues for free vibrations together with proper model formulations like the bound formulation (see [1,4] and the references therein).

Optimization problems involving eigenmodes, i.e. such that either cost or constraints depends on eigenmodes are much more scarce in the literature. However, in the last years several models requiring to optimize eigenmodes or constraining functionals depending on eigenmodes have been studied. In [5], a mode shape corresponding to a simple

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http://dx.doi.org/10.1016/j.cma.2017.07.031

eigenvalue of a fiber laser package is designed in order to minimize the elongation of the fiber under dynamic excitation. In [6], a multi-objective function is formulated in order to find optimal configurations that simultaneously satisfy (simple) eigenfrequency, eigenmode, and stiffness requirements at certain points of a vibrating structure. A similar problem is treated in [7]. The novelty there is to include the electromechanical coupling coefficient in the objective function so that the energy conversion is maximized for a specific mode. In [8], eigenmodes appear in the constraints only. One of the objectives of that work is to determine the material distribution of a structure that maximizes the fundamental frequency and at the same time synthesizes the first two modes. Recently, in [9] an optimization problem in which both cost and constraints depend on eigenmodes has been studied, and furthermore, all the pathologies appearing when dealing with eigenproblems are present: spurious modes, mode switching and multiple eigenfrequencies. The objective in that paper is the optimal design of piezoelectric modal sensor/actuators, in which we simultaneously designed the ground structure and polarization profiles of the piezoelectric transducers (see the recent survey [10]).

One of the main difficulties when dealing with eigenmode optimization is the sensitivity analysis. Eigenvector sensitivity for structural analysis problems has been addressed in [11–19]. The first issue is about differentiability, while eigenmodes corresponding to non-repeated eigenvalues are differentiable, eigenmodes corresponding to repeated ones are not even continuous. This is due to the fact that there are infinite eigenvector basis of the eigenspace associated to a multiple eigenvalue, so that this ambiguity spoils continuity and therefore differentiability. However, differentiability can be guaranteed for a concrete eigenvector basis, the so-called adjacent basis, and for it explicit derivative computations can be performed. Further to this, the eigenvector derivative computation leads to solve degenerate linear systems for which specific methods have been developed [11,13,14,16,17].

Another crucial point is eigenmode selection. When a functional involved in the optimization problem, either the cost or a constraint, depends on a mode shape it must be very clear on which specific mode shape is such a dependence, since mode shapes change and mode switching (change of order of eigenvalues in the spectrum) happens during the optimization process when updating variables. Therefore cost and constraints must clearly depend on *selected* mode shapes. In [9], a procedure for selecting the closest mode shapes to given reference mode shapes was developed. This procedure follows the idea and spirit of the well-known modal assurance criterion [20] but including the possibility of repeated eigenvalues appearance. The difficulty now is that once eigenmodes have been selected, the optimization algorithm requires derivatives for those specific eigenmodes. This is not an issue for eigenmodes associated with non-repeated eigenvalues since those are differentiable, but if they are associated with repeated ones, then available methods only provide derivatives for the adjacent basis of eigenvectors. For the specific situation considered by the authors in [9], a heuristic method working out for obtaining these derivatives was developed. The aim of this paper is to deepen those ideas whenever tracking specific mode shapes with three objectives in mind: establish a method of eigenmode selection suitable for a wide range of situations of eigenmode optimization models, give a general algorithm for computing derivatives of selected eigenmodes valid for the case of repeated and non-repeated eigenvalues, and finally, analyze the mathematical foundations of the algorithm in order to show its validity. These three objectives have been reached in this paper.

The paper is organized as follows. In Section 2 we review the main algorithms for eigenvector derivatives computation and comment on differentiability, clarifying some misleading ideas and claims present in the literature. In Section 3 we present a method for eigenmode selection and for the calculation of derivatives of selected eigenmodes. The algorithm validity is numerically corroborated with two structural analysis examples. Finally, in Section 4 we consider an example of an eigenmode optimization problem inspired by [9] in order to numerically validate the previous algorithm in a real optimization problem.

## 2. Review of current methods for eigenvectors sensitivity analysis

Given $\mathbf{K}, \mathbf{M} \in \mathbb{R}^{n\times n}$ real, symmetric and positive definite matrices. We are interested in the eigenproblem

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{U} = \mathbf{0}. \tag{1}$$

In structural dynamics, $\mathbf{K}$ stands for the stiffness matrix, $\mathbf{M}$ the mass matrix, $\lambda$ the eigenvalues or eigenfrequencies, i.e. the square of the natural frequencies, and $\mathbf{U}$ stands for the mode shape, or eigenvector, corresponding to the eigenfrequency $\lambda$.

It is well known that under these conditions all the eigenfrequencies are real, $\lambda_1 \leq \cdots \leq \lambda_n \in \mathbb{R}$, and there exists a $\mathbf{M}$-orthonormal basis of $\mathbb{R}^n$ made of eigenvectors $\mathbf{U}_1, \ldots, \mathbf{U}_n$, i.e.

$$(\mathbf{K} - \lambda_j\mathbf{M})\mathbf{U}_j = \mathbf{0} \tag{2}$$