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A Nitsche method for elliptic problems on composite surfaces[☆]

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Highlights

- A Nitsche method for problems on composite surfaces is presented.
- Each intersection curve may connect an arbitrary number of subsurfaces.
- The method fits nicely into isogeometric analysis as exact geometry is used.
- Non matching grids on each subsurface handled by adding stabilization terms.
- Error analysis independent of how the intersection curves cut the grid.

Abstract

We develop a finite element method for elliptic partial differential equations on so called composite surfaces that are built up out of a finite number of surfaces with boundaries that fit together nicely in the sense that the intersection between any two surfaces in the composite surface is either empty, a point, or a curve segment, called an interface curve. Note that several surfaces can intersect along the same interface curve. On the composite surface we consider a broken finite element space which consists of a continuous finite element space at each subsurface without continuity requirements across the interface curves. We derive a Nitsche type formulation in this general setting and by assuming only that a certain inverse inequality and an approximation property hold we can derive stability and error estimates in the case when the geometry is exactly represented. We discuss several different realizations, including so called cut meshes, of the method. Finally, we present numerical examples.

Keywords: Nitsche method; Composite surfaces; Laplace–Beltrami operator; A priori error estimates

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1. Introduction

Background

Many physical phenomena take place on geometries that consist of an arrangement of surfaces, for instance transport of surfactants, heat transfer, and flows in cracks. See Fig. 1 for examples of surface arrangements. In manufacturing the use of surface arrangements to minimize the amount of needed material, for example in the form of honeycomb sandwich structures, is well established while recent developments of additive manufacturing enable production of even more complex surface structures. The arrangement of surfaces in applications often contain sharp edges, corners, and lines where several surfaces meet. Thus there is significant interest in the development of finite element methods for solving partial differential equations on such general geometries.

New Contributions

In this contribution we develop a Nitsche method for a diffusion problem on a such an arrangement of surfaces. The key feature is that the formulation can handle interfaces where several surfaces meet at intersecting interfaces, including triple points and sharp edges. The method avoids defining a conormal to each interface and instead the well defined conormal associated with each subsurface is used together with the natural conservation law: the sum of all conormal fluxes is zero at the interface. This conservation law is sometimes referred to as the Kirchhoff condition. We show that the method is equivalent to the standard Nitsche interface method in flat geometries. The same idea naturally extends to discontinuous Galerkin methods on surfaces, where instead of defining a conormal to each edge which would be needed in a standard discontinuous Galerkin method, see for instance the discussion in [1], the well defined discrete element conormal can be used.

We consider different ways of constructing a mesh on a composite surface geometry including matching meshes, non matching meshes, and cut meshes. For cut meshes we add a stabilization term that provides control in the vicinity of the interfaces. More precisely we consider a stabilization term that satisfies certain abstract conditions which enable us to prove discrete stability and a priori error estimates independently of the explicit construction of the stabilization term. For clarity we restrict our analysis and numerical examples to the Laplace–Beltrami operator in the case when the geometry is exactly represented. This can be realized using parametric mappings, see [2]. We give a concrete construction of such a stabilization term based on penalization of jumps in derivatives across faces belonging to elements that intersect the interface, see [3] and [2], and we also suggest an alternative stabilization term which is not strongly consistent. In support of the analysis we also present some illustrating numerical examples.

Earlier Work

Since the pioneering work of Dziuk [4] where a continuous Galerkin method for the Laplace–Beltrami operator on a triangulated surface was first proposed there have been several extensions including adaptive and higher order methods, Demlow and Dziuk [5] and Demlow [6], and higher order problems, Clarenz et al. [7] and Larsson and Larson [8]. A standard discontinuous Galerkin method for the Laplace–Beltrami operator on a smooth closed surface was analyzed in [1]. For further extensions including time dependent problems we also refer to the review articles [9,10]. Models of membranes were considered in [11] and [12]. All of these contributions deal with smooth surfaces and the discontinuous Nitsche formulation proposed in this paper which allows more complex surfaces appears to be new. In [13] we develop a method for plate structures on composite surfaces consisting of plane surfaces with the restriction that only two plates meet at an interface. Here each plate is modeled using a membrane model and a fourth order Kirchhoff model for the bending. Various methods for connecting parametric patches pairwise which also allow for cut meshes have been proposed, see for example [14–16] and the extension to surfaces in [2].

Outline

The outline of the remainder of the paper is as follows. In Section 2 we give a short introduction to tangential calculus on surfaces and then we formulate the model problem. In Section 3 we derive the method, study how the interface terms in flat geometry relate to the average-jump terms used in standard discontinuous Galerkin methods, and introduce the stabilization term. In Section 4 we prove a priori error estimates in the energy and L^2 norm. In Section 5 we present some numerical examples illustrating the method on three different test cases and finally, in Section 6 we give some concluding remarks.