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Robust a posteriori error estimation for finite element approximation to $\boldsymbol{H}(\mathbf{curl})$ problem

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Highlights

- A novel a posteriori error estimator for $\boldsymbol{H}(\mathbf{curl})$ interface problem is invented.
- A new localized recovery procedure is proposed.
- The reliability is totally independent of the coefficients distribution.
- The estimator can handle irregular data that is not in $\boldsymbol{H}(\mathbf{div})$.

Abstract

In this paper, we introduce a novel a posteriori error estimator for the conforming finite element approximation to the $\boldsymbol{H}(\mathbf{curl})$ problem with inhomogeneous media and with the right-hand side only in $\boldsymbol{L}^2$. The estimator is of the recovery type. Independent with the current approximation to the primary variable (the electric field), an auxiliary variable (the magnetizing field) is recovered in parallel by solving a similar $\boldsymbol{H}(\mathbf{curl})$ problem. An alternate way of recovery is presented as well by localizing of the error flux. The estimator is then defined as the sum of the modified element residual and the residual of the constitutive equation defining the auxiliary variable. It is proved that the estimator is approximately equal to the true error in the energy norm without the quasi-monotonicity assumption. Finally, we present numerical results for several $\boldsymbol{H}(\mathbf{curl})$ interface problems.

MSC: 65N15; 65N30; 65N50

Keywords: Maxwell's equations; Nédélec finite elements; A posteriori error estimation; Interface problem; Flux recovery; Duality error estimation

1. Introduction

Let Ω be a bounded and simply-connected polyhedral domain in $\mathbb{R}^3$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$, and let $\boldsymbol{n} = (n_1, n_2, n_3)$ be the outward unit vector normal to the boundary. Denote by $\boldsymbol{u}$ the electric

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field, we consider the following $\boldsymbol{H}(\mathbf{curl})$ model problem originated from a second order hyperbolic equation by eliminating the magnetic field in Maxwell's equations:

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \boldsymbol{u}) + \beta\,\boldsymbol{u} = \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g}_D, & \text{on } \Gamma_D, \\ (\mu^{-1}\nabla \times \boldsymbol{u}) \times \boldsymbol{n} = \boldsymbol{g}_N, & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\nabla\times$ is the curl operator; the $\boldsymbol{f}$, $\boldsymbol{g}_D$, and $\boldsymbol{g}_N$ are given vector fields which are assumed to be well-defined on Ω, Γ_D, and Γ_N, respectively; the μ is the magnetic permeability; and the β depends on the electrical conductivity, the dielectric constant, and the time step size. Assume that the coefficients $\mu^{-1} \in \boldsymbol{L}^\infty(\Omega)$ and $\beta \in \boldsymbol{L}^\infty(\Omega)$ are bounded below

$$0 < \mu_0^{-1} \le \mu^{-1}(\boldsymbol{x}) \quad \text{and} \quad 0 < \beta_0 \le \beta(\boldsymbol{x})$$

for almost all $\boldsymbol{x} \in \Omega$.

The *a posteriori* error estimation for the conforming finite element approximation to the $\boldsymbol{H}(\mathbf{curl})$ problem in (1.1) has been studied recently by several researchers. Several types of *a posteriori* error estimators have been introduced and analyzed. These include residual-based estimators and the corresponding convergence analysis (explicit [1–7], and implicit [8]), equilibrated estimators [9], and recovery-based estimators [10,11]. There are four types of errors in the explicit residual-based estimator (see [1]). Two of them are standard, i.e., the element residual, and the interelement face jump induced by the discrepancy induced by integration by parts associated with the original equation in (1.1). The other two are also the element residual and the interelement face jump, but associated with the divergence of the original equation: $\nabla \cdot (\beta \boldsymbol{u}) = \nabla \cdot \boldsymbol{f}$, where $\nabla\cdot$ is the divergence operator. These two quantities measure how good the approximation is in the kernel space of the curl operator.

Recently, the idea of the robust recovery estimator explored in [12,13] for the diffusion interface problem has been extended to the $\boldsymbol{H}(\mathbf{curl})$ interface problem in [10]. Instead of recovering two quantities in the continuous polynomial spaces like the extension of the popular Zienkiewicz–Zhu (ZZ) error estimator in [11], two quantities related to $\mu^{-1}\nabla \times \boldsymbol{u}$ and $\beta\boldsymbol{u}$ are recovered in the respective $\boldsymbol{H}(\mathbf{curl})$- and $\boldsymbol{H}(\text{div})$-conforming finite element spaces. The resulting estimator consists of four terms similar to the residual estimator in the pioneering work [1] on this topic by Beck, Hiptmair, Hoppe, and Wohlmuth: two of them measure the face jumps of the tangential components and the normal component of the numerical approximations to $\mu^{-1}\nabla \times \boldsymbol{u}$ and $\beta\boldsymbol{u}$, respectively, and the other two are element residuals of the recovery type.

All existing a posteriori error estimators for the $\boldsymbol{H}(\mathbf{curl})$ problem assume that the right-hand side $\boldsymbol{f}$ is in $\boldsymbol{H}(\text{div})$ or divergence free. This assumption does not hold in many applications (e.g. the implicit marching scheme mentioned in [14]). Moreover, two terms of the estimators are associated with the divergence of the original equation. In the proof, these two terms come to existence up after performing the integration by parts for the irrotational gradient part of the error, which lies in the kernel of the curl operator. One of the key technical tools, a Helmholtz decomposition, used in this proving mechanism, relies on $\boldsymbol{f}$ being in $\boldsymbol{H}(\text{div})$, and fails if $\boldsymbol{f} \notin \boldsymbol{H}(\text{div})$. In [4], the assumption that $\boldsymbol{f} \in \boldsymbol{H}(\text{div})$ is weakened to $\boldsymbol{f}$ being in the piecewise $\boldsymbol{H}(\text{div})$ space with respect to the triangulation, at the same time, the divergence residual and norm jump are modified to incorporate this relaxation. Another drawback of using Helmholtz decomposition on the error is that it introduces the assumption of the coefficients' quasi-monotonicity into the proof pipeline. An interpolant with a coefficient independent stability bound is impossible to construct in a "checkerboard" scenario (see [15] for diffusion case, and [10] for $\boldsymbol{H}(\mathbf{curl})$ case). To gain certain robustness for the error estimator in the proof, one has to assume the coefficients distribution is quasi-monotone. However, in an earlier work of Chen, Xu, and Zou [3], it is shown that numerically this quasi-monotonicity assumption is more of an artifact introduced by the proof pipeline, at least for the irrotational vector fields. As a result, we conjecture that the divergence related terms should not be part of an estimator if it is appropriately constructed. In Section 5, some numerical justifications are presented to show the unnecessity of including the divergence related terms.

The pioneering work in using the dual problems for a posteriori error estimation dates back to [16]. In [16], Oden, Demkowicz, Rachowicz, and Westermann studied the a posteriori error estimation through duality for the diffusion–reaction problem. The finite element approximation to a dual problem is used to estimate the error for the original primal problem (diffusion–reaction). The result shares the same form to the Prager–Synge identity [17] for diffusion–reaction problem. The method presented in this paper may be viewed as an extension of the duality method