Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 309 (2016) 579–609
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

The DPG methodology applied to different variational formulations of linear elasticity

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Received 29 January 2016; received in revised form 27 May 2016; accepted 27 May 2016
Available online 23 June 2016

Abstract

The flexibility of the DPG methodology is exposed by solving the linear elasticity equations under different variational formulations, including some with non-symmetric functional settings (different infinite-dimensional trial and test spaces). The family of formulations presented is proved to be mutually ill or well-posed when using traditional energy spaces on the whole domain. Moreover, they are shown to remain well-posed when using broken energy spaces and interface variables. Four variational formulations are solved in 3D using the DPG methodology. Numerical evidence is given for both smooth and singular solutions and the expected convergence rates are observed.
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Keywords: Linear elasticity; DPG; Variational formulations; Adaptivity; Minimum residual method

1. Introduction

In this paper we demonstrate the fitness of the DPG finite element method with optimal test spaces on various variational formulations of the nondimensionalized equations of linear elasticity,

$$\begin{aligned} -\operatorname{div}(\mathsf{C} : \varepsilon(u)) &= f, && \text{in } \Omega, \\ u &= u_0, && \text{on } \Gamma_0, \\ (\mathsf{C} : \varepsilon(u)) \cdot \mathfrak{n} &= g, && \text{on } \Gamma_1. \end{aligned} \tag{1.1}$$

We take Ω to be a simply connected smooth domain in $\mathbb{R}^3$ and let Γ_0 and Γ_1 be a partition of the boundary, $\overline{\Gamma_0 \cup \Gamma_1} = \partial\Omega$ with outward unit normal, $\mathfrak{n}$. Here, u is the displacement, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ is the associated strain, f is the body force, g is the traction,[1] and u_0 is the prescribed displacement. Meanwhile, $\mathsf{C} : \mathbb{S} \to \mathbb{S}$, is the elasticity or stiffness tensor, where $\mathbb{S}$ denotes all symmetric 3×3 matrices. For isotropic materials, it is expressed as $\mathsf{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ are the Lamé parameters.

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[1] If $\overline{\Gamma_1} = \partial\Omega$, then f and g must satisfy Signorini's compatibility condition $\int_\Omega f \cdot v \, d\Omega + \int_{\Gamma_1} g \cdot v \, d\Gamma = 0$ for all infinitesimal rigid displacements, v.

http://dx.doi.org/10.1016/j.cma.2016.05.034
0045-7825/Published by Elsevier B.V.

It can be shown that the standard Bubnov–Galerkin finite element method for linear elasticity computes the unique minimizer of the energy functional $\mathcal{E}_1(v) = \int_\Omega (\frac{1}{2}\varepsilon(v) : \mathsf{C} : \varepsilon(v) - f \cdot v)\,\mathrm{d}\Omega - \int_{\Gamma_1} g \cdot v\,\mathrm{d}\Gamma$, over all candidates, v, in a discrete space of displacements, U_h. By proceeding from an energy minimization we guarantee to compute the best possible solution (measured in the energy) allowed in our set of computable solutions (trial space). In this sense, the formulation has the obvious desirable quality that there is a meaningful metric of solution relevance as measured by the energy functional. The typical approach in commercial software is to use exactly the standard Bubnov–Galerkin variational formulation to simulate and predict elastic behavior in materials.

Notwithstanding the above method, there are important circumstances where such a simple energy minimization principle is avoided. Another prominent variational formulation for linear elasticity is the well-known mixed method [1]. These discretizations stem from the minimax problem on the Hellinger–Reissner energy functional $\mathcal{E}_2(\tau, v) = -\int_\Omega \big(\frac{1}{2}\tau : \mathsf{C}^{-1} : \tau + \mathrm{div}\tau \cdot v + f \cdot v\big)\,\mathrm{d}\Omega + \int_{\Gamma_0} u_0 \cdot (\tau \cdot \mathfrak{n})\,\mathrm{d}\Gamma$ [2], an energy principle equivalent to minimization of $\mathcal{E}_1$ [3,4]. Here v is a displacement variable and $\tau = \tau^{\mathsf{T}}$ is a stress variable. Such a formulation results in a discretization which avoids volumetric locking and also guarantees a locally conservative stress tensor [1]. Of course, this formulation also guarantees a best possible solution although it is measured in a different way and the trial spaces differ.

Likewise, other energy principles exist for linear elasticity problems. In fact, just for this single problem a total of 14 complementary-dual energy principles are presented in [5], each leading to a different variational formulation. Some may not be easily amenable to computation but perspective should be given that there is little to regard as sacred or more physical about one formulation over another. Ultimately, whatever the physical principle (energy functional) employed, the equations of linear elasticity are ubiquitous; beyond their functional setting, they do not change even though they can be derived in different ways and posed over different spaces. In principle, at the infinite-dimensional level the solution will always be the same but at the computational level the differences can become very important.

In the DPG method, we do not make a quandary over the best physical principle to employ for our choice of optimality. Instead, without access to the exact solution outright, we seek the best numerical solution available to us once the trial space and variational formulation are set. This is achieved by considering a minimization problem on the residual of the discrete solution taken through a user-defined norm in the discrete test space. The ramifications of this methodology are substantial, however analyzing most of them is not the particular focus of this paper. Instead, we intend only to demonstrate the utility of the methodology on various variational formulations. We will now outline some of the history and recent developments of DPG.

The optimal stability DPG methodology [6,7], referred here simply as "DPG", was originally envisioned as a practical Petrov–Galerkin finite element method which would naturally transfer the stability of the infinite dimensional problem onto the discrete system. This is achieved by exploiting a natural isometry between a Hilbert space and its dual, called the Riesz map, and the ability to localize its computation by using broken test spaces. In a difficult problem, instead of tuning stability parameters as is commonplace in standard stabilized methods, the DPG method algorithmically approximates an *optimal test space* to a tunable accuracy in a way that applies to all well-posed variational problems. The tuning parameter in the DPG method is usually the order of a local test space called the *enriched test space* where the Riesz map (in the user-defined norm) is computed. The larger this parameter, the more accurate the approximation of the optimal test space. For every computation in this paper we found it sufficient to choose an enriched test space one order larger than the trial space. Using a larger enrichment may not be a great hindrance, because the feasibility of the method is offered from the fact that all computations on this higher order enriched space are localized. Therefore, provided the element-local computations have been distributed (which can be done in parallel) and are made efficiently, the computational cost of the method is essentially independent of the enrichment parameter. However, each element-local computation can sometimes be computationally intensive if the enrichment parameter is too high. In this context, the choice of the user-defined norm of the test space can play a fundamental role in efficiently obtaining a well-approximated optimal test space while only requiring a modest enrichment parameter.

DPG distinguishes the trial and test space differently from the outset and because of this trait it is applicable to often neglected, non-symmetric variational formulations. This originally led to the DPG method with ultraweak variational formulations, a formulation wherein the trial space is naturally discontinuous. Some highlights of ultraweak variational formulations are given in [8–11]. Indeed, in this setting, DPG has largely been applied to singular perturbation problems and other problems in computational mechanics where stability is difficult to achieve such as advection