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Variational formulation and isogeometric analysis for fourth-order boundary value problems of gradient-elastic bar and plane strain/stress problems

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Abstract

The fourth-order boundary value problems of one parameter gradient-elastic bar and plane strain/stress models are formulated in a variational form within an H^2 Sobolev space setting. For both problems, the existence and uniqueness of the solution is established by proving the continuity and coercivity of the associated symmetric bilinear form. For completeness, the full sets of boundary conditions of the problems are derived and, in particular, the new types of boundary conditions featured by the gradient-elastic models are given the additional attributes *singly* and *doubly*. By utilizing the continuity and coercivity of the continuous problems, corresponding error estimates are formulated for conforming Galerkin formulations. Finally, numerical results, with isogeometric C^{p-1}-continuous discretizations for NURBS basis functions of order $p \geq 2$, confirm the theoretical results and illustrate the essentials of both static and vibration problems.

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Keywords: Gradient elasticity; Plane strain/stress; Variational formulation; Existence and Uniqueness; Isogeometric analysis; Error estimates

1. Introduction

Classical continuum theories, such as theories of linear or nonlinear elasticity and plasticity, have been widely used in various fields of science and engineering for modelling solids and structures. The ability of classical continuum theories for describing multi-scale phenomena is very limited, however. On the other hand, theories and methods for studying small-scale phenomena, such as molecular dynamics, are often inefficient in many applications eventually ruled by macro-scale conservations laws. This has raised a motivation for further development of single-scale continuum mechanics which has been extended towards multi-scale capabilities still preserving the most characteristic advantages of their homogenizing nature [1–3].

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In the current work, we concentrate on a single-parameter simplified theory (proposed in [4,5] in the 1990s) of the gradient elasticity theory of Form II derived by Mindlin (in the landmark paper [6] in the 1960s). However, our theoretical results can be extended to more general multi-parameter models of Form II and I in a natural way. Regarding experimental background and applications of gradient elasticity theories, we refer to the discussion in [7–9]. For determining the material parameters of the models, we refer to [10,7,11].

Within the classical elasticity theory, the engineering bar model as well as plane stress and plane strain models lead to second-order partial differential equations. Within the theory of gradient elasticity, instead, these models lead to fourth-order governing equations [12,13]. In particular, this fact complicates the numerical methods applied for solving the corresponding boundary value problems since higher order derivatives in Euler equations imply higher order (weak) derivatives in the corresponding variational formulations. Accordingly, the related trial and test functions of Galerkin methods, for instance, have to meet higher regularity conditions. For fourth-order Euler equations, in particular, C^1 continuity is required for satisfying the H^2 regularity in the weak form. Isogeometric analysis in the Galerkin sense (initiated in [14]) which provides straightforward C^{p-1} discretizations, for NURBS basis functions of order $p \geq 2$, appears as an even more attractive method for higher-order boundary value problems (see [15–18], for instance) than for lower order formulations of the classical elasticity theory (as [19,20], for instance).

Solvability of gradient-elastic problems has been addressed in few studies only, as in [5]. Uniqueness of the solution, in particular, has been discussed in the context of analytical solutions for a one-dimensional model problem in [21,22] by introducing the term “sign paradox”. The discussion on this issue has been extended in [7] for a class of more general constitutive tensors of gradient elasticity. A more theoretical analysis has been accomplished in [23] for solvability of two parameter fourth-order gradient elasticity problems written as a mixed variational formulation in an H^1 Sobolev space framework in order to utilize the Babuska–Brezzi theory. In this paper, we focus on the variational formulation and its solvability, i.e., existence and uniqueness, of the gradient-elastic bar and plane strain/stress problems corresponding to the fourth-order boundary value problems presented in [21,12,13,24,3]. Accordingly, we prove the well-posedness of the corresponding displacement based variational formulations within H^2 Sobolev space settings. In contrast to classical problems governed by fourth order partial differential equations, such as the Kirchhoff plate problem commanded by the biharmonic equation, problems of gradient elasticity are essentially affected by the gradient parameters associated to the higher order derivatives. Our proofs are established for clamped boundaries but they can be extended to other boundary condition types as well. Regarding boundary conditions, in particular, this paper recalls and identifies two different types of clamped and free boundary conditions and gives them additional attributes *singly* and *doubly*. Furthermore, corner conditions often omitted in the literature are derived as well. Altogether, our formalism provides a consistent framework for both variational and non-variational general-purpose numerical approximation methods.

Regarding numerical methods for gradient elasticity and applications, only a limited number of model problems with simple geometries and non-general boundary conditions has been solved, and mostly by analytical means (see [12,24,13,3,25,26]). In particular, for numerical methods capable of solving general geometries and different boundary conditions, the literature is very limited, even for bar and plane problems. First, in [27], a group of C^0-continuous elements based on a mixed formulation has been proposed; second, in [28], a consistent and stable discontinuous Galerkin method has been formulated, theoretically analysed and verified for a shear layer problem of strain gradient elasticity (formally identical to the bar problem); third, in [29,30], a couple of C^1-continuous elements has been introduced; fourth, in [31], a C^0-continuous approach has been applied. More recently, [16] has benchmarked static plane gradient elasticity problems by C^{p-1}-continuous isogeometric discretizations, with $p \geq 2$ referring to the order of NURBS basis functions. Finally, in [32], three-dimensional problems of Toupin’s gradient elasticity theory at finite strains have been solved by applying H^2-conforming, i.e., at least C^1-continuous, isogeometric discretizations, including the one-dimensional bar problem with infinitesimal strains as a verification benchmark for convergence studies. All of the aforementioned articles concentrate on the static problems of gradient elasticity. Furthermore, only [28] concentrates on formulating and analysing the problems and related methods within a theoretical framework. In this paper, we fill this gap present on the theoretical side for the continuous formulations and the corresponding conforming Galerkin methods for both bar and plane gradient elasticity problems. In addition, our numerical benchmarks and examples confirm the theoretical results and clarify the effect of the gradient terms on both static and vibration problems, the latter being practically impossible to analyse without a numerical approach for plane strain/stress problems.

This paper is organized as follows: In Section 2, we introduce our notation by recalling the variational formulations of the gradient-elastic bar model as well as plane stress and plane strain models. For completeness, in Section 3,