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Fractional-order uniaxial visco-elasto-plastic models for structural analysis

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Highlights

- Two fractional-order visco-elasto-plastic models are developed, namely M1 and M2.
- Model M1 uses a rate-dependent yield function via time-fractional Caputo derivative.
- Model M2 uses a visco-plastic regularization via time-fractional Caputo derivative.
- A fractional return-mapping algorithm is developed for each model.
- Results show flexibility of the fractional-orders and recovery of classical models.

Abstract

We propose two fractional-order models for uniaxial large strains and visco-elasto-plastic behavior of materials in structural analysis. Fractional modeling seamlessly interpolates between the standard elasto-plastic and visco-elasto-plastic models, taking into account the history (memory) effects of the accumulated plastic strain to specify the state of stress. To this end, we develop two models, namely M1 and M2, corresponding to visco-elasto-plasticity considering a rate-dependent yield function and visco-plastic regularization, respectively. Specifically, we employ a fractional-order constitutive law that relates the Kirchhoff stress to the Caputo time-fractional derivative of the strain with order $\beta \in (0, 1)$. When $\beta \to 0$ the standard rate-independent elasto-plastic model with linear isotropic hardening is recovered by the models for general loading, and when $\beta \to 1$, the corresponding classical visco-plastic model of Duvaut–Lions (Perzyna) type is recovered by the model M2 for monotonic loading. Since the material behavior is path-dependent, the evolution of the plastic strain is achieved by fractional-order time integration of the plastic strain rate with respect to time. The plastic strain rate is then obtained by means of the corresponding plastic slip and proper consistency conditions. Finally, we develop the so called *fractional return-mapping algorithm* for solving the nonlinear system of the equilibrium equations developed for each model. This algorithm seamlessly generalizes the standard return-mapping algorithm to its fractional counterpart. We test both models for convergence subject to prescribed strain rates, and subsequently we implement the models in a finite element truss code and solve for a two-dimensional snap-through instability problem. The simulation results demonstrate the flexibility of fractional-order modeling using the Caputo derivative to account for rate-dependent hardening and

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viscous dissipation, and its potential to effectively describe complex constitutive laws of engineering materials and especially biological tissues.

Keywords: Fractional-order constitutive laws; History-dependent visco-elasto-plasticity; Large strains; Time-fractional integration

1. Introduction

Fractional differential operators appear in many systems in science and engineering such as visco-elastic materials [1–3], electrochemical processes [4] and porous or fractured media [5]. For instance, it has been found that the transport dynamics in complex and/or disordered systems is governed by non-exponential relaxation patterns [6,7]. For such processes, a time-fractional equation, in which the time-derivative emerges as $\mathcal{D}_t^{\nu} u(t)$, appears in the continuous limit. One interesting application of fractional calculus is to model complex elasto-plastic behavior of engineering materials (e.g. [8,9]). Recently, fractional calculus has been employed as an effective tool for modeling materials accounting for heterogeneity/multi-scale effects to the constitutive model [10–12], where the fractional visco-plasticity was introduced as a generalization of classical visco-plasticity of Perzyna type [13]. The fundamental role of the formulation is the definition of the flow rule by introducing a fractional gradient of the yield function. Also, a constitutive model for rate-independent plasticity based on a fractional continuum mechanics framework accounting for nonlocality in space was developed in [14].

Formulating fast and accurate numerical methods for solving the resulting system of fractional ODEs/PDEs in such problems is the key to incorporating such nonlocal/history-dependent models in engineering applications. Efficient discretization of the fractional operators is crucial. Lubich [15,16] pioneered the idea of *discretized fractional calculus* within the spirit of finite-difference method (FDM). Later, Sanz-Serna [17] adopted the idea of Lubich and presented a temporal semi-discrete algorithm for partial integro-differential equations, which was first order accurate. Sugimoto [18] also employed a FDM for approximating the fractional derivative emerging in Burgers' equation. Later on, Gorenflo et al. [19] adopted a finite-difference scheme by which they could generate discrete models of random walk in such anomalous diffusion. Diethelm et al. proposed a predictor–corrector scheme in addition to a fractional Adams method [20,21]. After that, Langlands and Henry [22] considered the fractional diffusion equation, and analyzed the L^1 scheme for the time-fractional derivative. Sun and Wu [23] also constructed a difference scheme with L^{∞} approximation of the time-fractional derivative. Of particular interest, Lin and Xu [24] analyzed a FDM for the discretization of the time-fractional diffusion equation with order $(2-\alpha)$. However, there are other classes of global methods (spectral and spectral element methods) for discretizing fractional ODEs/PDEs (e.g., [25–28]), which are efficient for low-to-high dimensional problems. Furthermore, Zayernouri and Matzavinos developed a fractional family of schemes, called *fractional Adams–Bashforth and fractional Adams–Moulton* method for high-order explicit and implicit treatment of nonlinear problems [29]. There were recent developments on meshless approaches applied to fractional-diffusion and space-fractional advection–dispersion problems [30,31]. Also, Chen [32] developed a new definition of fractional Laplacian and applied to three-dimensional, nonlocal heat conduction.

The main contribution of the present work is to propose and solve two fractional-order models, namely M1 and M2, for uniaxial large strains and visco-elasto-plastic behavior of materials. Both models account for fractional visco-elastic modeling by defining a stress–strain relationship involving the Caputo time derivative of fractional-order, but have distinct formulations to model the fractional visco-plasticity. For the model M1, visco-plasticity is achieved by including history effects in time for the internal hardening parameter in the yield function, making it rate-dependent. Differently from some works found in the literature [10–12], we do not modify the flow rule. The model M2 accounts for a rate-independent yield function without an internal hardening parameter, and the visco-plastic effect is achieved based on the approach of *visco-plastic regularization* used in the classical visco-plastic model of Duvaut–Lions type (which is equivalent to Perzyna's model). Furthermore, the models consider different memory effects for visco-elasticity and visco-plasticity. Both models are used within the framework of a time-fractional backward-Euler integration procedure with a *fractional return-mapping algorithm*, based on the classical models in the literature [9,8]. The developed algorithm seamlessly generalizes the standard return-mapping algorithm to its fractional counterpart, making it amenable for path-dependent visco-elasto-plastic analyses in engineering and bio-