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# A cut finite element method for coupled bulk-surface problems on time-dependent domains

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**Abstract**

In this contribution we present a new computational method for coupled bulk-surface problems on time-dependent domains. The method is based on a space–time formulation using discontinuous piecewise linear elements in time and continuous piecewise linear elements in space on a fixed background mesh. The domain is represented using a piecewise linear level set function on the background mesh and a cut finite element method is used to discretize the bulk and surface problems. In the cut finite element method the bilinear forms associated with the weak formulation of the problem are directly evaluated on the bulk domain and the surface defined by the level set, essentially using the restrictions of the piecewise linear functions to the computational domain. In addition a stabilization term is added to stabilize convection as well as the resulting algebraic system that is solved in each time step. We show in numerical examples that the resulting method is accurate and stable and results in well conditioned algebraic systems independent of the position of the interface relative to the background mesh.


*Keywords:* Coupled bulk-surface problems; Cut finite element method (CutFEM); Space–time FEM; Evolving domains; Soluble surfactant; Sharp interface method

## 1. Introduction

Problems involving phenomena that take place both on surfaces (or interfaces) and in bulk domains occur in a variety of applications in fluid dynamics and biology. In this paper, we consider a coupled bulk-surface problem modeling the evolution of soluble surfactants. A soluble surfactant is dissolved in the bulk fluid but also exists in adsorbed form on the interface separating two immiscible fluids. Surfactants have a large influence on the dynamics in multiphase flow systems in that they may cause drop-breakup or coalescence due to their ability to reduce the surface tension. They were for example used to lower the surface tension of oil droplets in the 2010 Deepwater Horizon oil

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spill so that the oil became more soluble in the water. Other examples of applications where the effects of surfactants are important include drug delivery, treatment of lung diseases, and polymer blending [1].

We consider a coupled system of time-dependent convection–diffusion equations describing the concentration of surfactants in the bulk fluid and on the interface. The interface is moving with a given velocity. From a computational point of view, the main challenge is that the differential equations are defined on domains that are evolving with time and that these domains may undergo strong deformations.

A common strategy is to let the mesh conform to the time-dependent domain, see, e.g., [2,3]. This technique can be made accurate but requires remeshing and interpolation as an interface evolves with time and leads to significant complications when topological changes such as drop-breakup or coalescence occur, especially in three space dimensions. Therefore, computational methods that allow the interface to be arbitrarily located with respect to a fixed background mesh, so called fixed grid methods, have become highly attractive and significant effort has been directed to their development, see, e.g., [4–7]. In fixed grid methods a strategy for solving the bulk Partial Differential Equation (PDE) defined on a domain with the interface as boundary is to extend the PDE to the whole computational domain by for example regularized characteristic functions, cf. [8,9]. Strategies for solving quantities on evolving surfaces are in general developed on the basis of the interface representation technique. In consequence, existing fixed grid methods are usually tightly coupled to the interface representation. Techniques to represent the interface can be roughly divided into two classes: *explicit* representation, e.g., by marker particles [10], and *implicit* representation, e.g., by the level set of a higher dimensional function [11]. Existing methods using implicit representation techniques generally extend the surfactant concentration to a region embedding the interface, and instead of a surface PDE, a PDE on a higher dimensional domain must be solved for the interfacial surfactant. Several methods have been proposed, based on explicit [9,12–16] as well as implicit [17–21] representations. Most work has been done on insoluble surfactants, i.e., surfactants that are only present at the interface without surfactant mass transfer between the interface and the bulk.

In this paper, we present a new computational method for solving coupled bulk-surface problems on time-dependent domains. The surface PDE is solved on the interface which can be arbitrarily located with respect to the fixed background mesh. The method is accurate and stable and results in well conditioned linear systems independently of how the interface cuts through the background mesh, and the total mass of surfactants is accurately conserved.

Our strategy is to embed the time-dependent domain where the PDE has to be solved in a fixed background grid, equipped with a standard finite element space, and then take the restriction of the finite element functions to the time-dependent domain. This idea was first proposed for an elliptic problem with a stationary fictitious boundary in [7] and for the Laplace–Beltrami operator on a stationary interface in [22]. It has been extended to other equations with error analysis, for example the Stokes equations involving two immiscible incompressible fluids with different viscosities and with surface tension [23], to PDE:s on time-dependent surfaces in, e.g., [24–27], and to stationary coupled bulk-surface problems with linear coupling terms in [28,29]. These types of methods are referred to as cut finite element methods (CutFEM), since the interface cuts through the background grid in an arbitrary fashion.

We suggest a CutFEM based on a space–time approach with continuous linear elements in space and discontinuous piecewise linear elements in time. The method presented in [25] is for solving surface PDEs but is also based on a space–time approach with discontinuous elements in time. However, in our approach we add a consistent stabilization term [30,31] which ensures that (1) our method leads to linear systems with bounded condition number, (2) the discretization of the surface PDE is stable also for convection dominated problems, and (3) the proposed method relies only on spatial discretizations of the geometry at quadrature points in time and results in the same computations as in the case of stationary problems. In addition, the total mass of surfactants is accurately conserved using a Lagrange multiplier. Numerical results indicate that the method is optimal order accurate (second order); we have also proven optimal order of accuracy for a related stationary coupled bulk-surface problem with a linear coupling term in [28].

In this paper, we have used the standard level set method to represent the interface, but other interface representation techniques can be used as well. We have chosen to concentrate on the challenging task of solving the coupled bulk-surface problem on time-dependent domains and will throughout the paper assume that the velocity field is given.

The remainder of the paper is outlined as follows. In Section 2 we formulate the coupled bulk surface problem. In Section 3 we introduce a discrete approximation of the interface and state our assumptions on the geometry. The