Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 307 (2016) 362–382
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A new stabilized linear finite element method for solving reaction–convection–diffusion equations

Po-Wen Hsieh[a], Suh-Yuh Yang[b,*]

[a] Department of Applied Mathematics, Chung Yuan Christian University, Jhongli District, Taoyuan City 32023, Taiwan
[b] Department of Mathematics, National Central University, Jhongli District, Taoyuan City 32001, Taiwan

Received 28 August 2014; received in revised form 18 April 2016; accepted 22 April 2016
Available online 6 May 2016

Abstract

In this paper, we propose a new stabilized linear finite element method for solving reaction–convection–diffusion equations with arbitrary magnitudes of reaction and diffusion. The key feature of the new method is that the test function in the stabilization term is taken in the adjoint-operator-like form $-\varepsilon\Delta v-(\boldsymbol{a}\cdot\nabla v)/\gamma+\sigma v$, where the parameter γ is appropriately designed to adjust the convection strength to achieve high accuracy and stability. We derive the stability estimates for the finite element solutions and establish the explicit dependence of L^2 and H^1 error bounds on the diffusivity, modulus of the convection field, reaction coefficient and the mesh size. The analysis shows that the proposed method is suitable for a wide range of mesh Péclet numbers and mesh Damköhler numbers. More specifically, if the diffusivity ε is sufficiently small with $\varepsilon<\|\boldsymbol{a}\|h$ and the reaction coefficient σ is large enough such that $\|\boldsymbol{a}\|<\sigma h$, then the method exhibits optimal convergence rates in both L^2 and H^1 norms. However, for a small reaction coefficient satisfying $\|\boldsymbol{a}\|\geq\sigma h$, the method behaves like the well-known streamline upwind/Petrov–Galerkin formulation of Brooks and Hughes. Several numerical examples exhibiting boundary or interior layers are given to demonstrate the high performance of the proposed method. Moreover, we apply the developed method to time-dependent reaction–convection–diffusion problems and simulation results show the efficiency of the approach.

MSC: 65N12; 65N15; 65N30

Keywords: Reaction–convection–diffusion equation; Boundary layer; Interior layer; Stabilized finite element method; Stabilization parameter

1. Introduction

In this paper, we are interested in the stabilized linear finite element approximations to the following Dirichlet boundary value problem for the reaction–convection–diffusion equation:

$$\begin{cases}-\varepsilon\Delta u+\boldsymbol{a}\cdot\nabla u+\sigma u=f & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases}\tag{1}$$

* Corresponding author. Tel.: +886 3 4227151x65130; fax: +886 3 4257379.
E-mail addresses: pwhsieh0209@gmail.com (P.-W. Hsieh), syyang@math.ncu.edu.tw (S.-Y. Yang).

http://dx.doi.org/10.1016/j.cma.2016.04.024

where $\Omega \subset \mathbb{R}^2$ is an open bounded convex polygonal domain with boundary $\partial\Omega$, u is the physical quantity of interest (e.g., temperature in heat conduction or concentration of some chemical substance), $0 < \varepsilon \le 1$ is the constant diffusivity, $\boldsymbol{a} \in (H^1(\Omega) \cap L^\infty(\Omega))^2$ is the given convection (velocity) field satisfying $\nabla \cdot \boldsymbol{a} = 0$ in Ω, $\sigma \ge 0$ is the constant reaction coefficient, and $f \in L^2(\Omega)$ is the given source function. It is well known that when the diffusivity ε is relatively small compared with the modulus of the convection field $\boldsymbol{a}$ or the reaction coefficient σ, the solution u of problem (1) may exhibit localized phenomena such as boundary and interior layers [1–5]. Boundary and interior layers are some narrow regions in the immediate vicinity of the domain boundary $\partial\Omega$ or in the interior of the domain Ω where the solution has large gradients. It is often difficult to numerically resolve the solution within the neighborhood of the layer regions, and the conventional numerical methods usually produce low accuracy or suffer from instability [1–5]. For instance, the standard Galerkin method using continuous piecewise linear (P_1) or bilinear (Q_1) elements performs very poorly since large spurious oscillations exhibit not only in the layer regions but also in other regions. Therefore, in the past three decades, a large class of the so-called stabilized finite element methods (FEMs) has been intensively developed to overcome this difficulty, see, e.g., [6–13]. The stabilized FEMs are formed by adding to the standard Galerkin method some consistent variational terms, relating to the residuals of the partial differential equations, which involve some mesh-dependent stabilization parameters. A robust approach for the derivation of such a stabilized FEM is motivated by the bubble-enriched method [14–17] combined with the procedure of static condensation [18]. It is now clear that the stabilization parameters play the key roles in the stabilization method. To a great degree, they account for why those additional stabilization terms not only can enhance the numerical stability but also can improve the accuracy in the finite element solutions.

In this paper, we will focus on developing efficient stabilized FEMs for solving the reaction–convection–diffusion problem (1). First, let us give a brief review of some previous works, which are closely related with the new method that we will introduce in this paper. In [18], Franca and Farhat proposed a so-called unusual stabilized linear FEM for problem (1) with vanishing convection field $\boldsymbol{a}$. They proved that the error estimate is optimal in H^1-seminorm independent of the values of ε and σ. In addition, for $\varepsilon \le \sigma h_T^2$ for all elements, optimal order in L^2 norm can also be obtained without using the duality argument. They also considered the problem (1) including the convection term $\boldsymbol{a} \cdot \nabla u$ and suggested a stabilization parameter to deal with all the three effects from reaction, convection and diffusion simultaneously, but no analysis is given therein. In [19], Franca and Valentin constructed a new stabilization parameter for the presence of the convection term to improve the accuracy. The improvement is also justified therein from an error analysis. Some further results have also been achieved by Duan [20]. In [21], Hauke, Sangalli and Doweidar proposed an efficient stabilized FEM for solving the reaction–convection–diffusion problems. Their method combines two types of stabilization integrals, namely an adjoint stabilization and a gradient adjoint stabilization, and two stabilization parameters are involved therein. These two parameters are chosen based on imposing one-dimensional nodal exactness. More recently, we devised a new stabilized FEM for problem (1) in [22], with emphasis on the case of small diffusivity ε and large reaction coefficient σ. As usual, we employed the continuous piecewise P_1 (or Q_1) elements and used the residual of the differential equation in problem (1) to define the stabilization term, but in which a novel stabilization parameter is carefully designed. The main differences from the stabilization methods proposed in [18] and [19] are that the stabilization parameter is deterministic and explicit, without the comparisons among the three effect-terms: reaction, convection and diffusion; the stabilization parameter is always the same no matter if the convection $\boldsymbol{a}$ is present or not in problem (1); and the test function involved in the stabilization term is taken in the form $-\varepsilon\Delta v + \sigma v$, instead of the adjoint-operator form $-\varepsilon\Delta v - \boldsymbol{a} \cdot \nabla v + \sigma v$ in [18] and [19]. The stabilized linear FEM proposed in [22] has been proved to be very effective for problem (1) with a small diffusivity ε and a large reaction coefficient σ.

In this paper, we will propose a new stabilized linear FEM for solving reaction–convection–diffusion equations with arbitrary magnitudes of reaction and diffusion. The key feature of the new method is that the test function in the stabilization term is taken in the adjoint-operator-like form $-\varepsilon\Delta v - (\boldsymbol{a} \cdot \nabla v)/\gamma + \sigma v$, where the stabilization parameter γ will be appropriately designed to adjust the convection strength to achieve high accuracy and stability. We will explicitly establish the dependence of L^2 and H^1 error bounds on the diffusivity ε, modulus of the convection field, given as $\|\boldsymbol{a}\| := \operatorname{ess\,sup}_{(x,y)\in\Omega}(a_1^2(x, y) + a_2^2(x, y))^{1/2}$, reaction coefficient σ and the mesh size h. Our analysis shows that the proposed method is suitable for a wide range of mesh Péclet numbers, defined as $Pe_h := \|\boldsymbol{a}\|h/(2\varepsilon)$, and mesh Damköhler numbers, given by $Da_h := \sigma h/\|\boldsymbol{a}\|$. More specifically, if the diffusivity ε is sufficiently small with $\varepsilon < \|\boldsymbol{a}\|h$ (i.e., $Pe_h > 1/2$) and the reaction coefficient σ is large enough such that $\|\boldsymbol{a}\| < \sigma h$ (i.e., $Da_h > 1$), then the proposed method exhibits optimal convergence rates in both L^2 and H^1 norms, with respect to the mesh size