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A paradigm for higher-order polygonal elements in finite elasticity using a gradient correction scheme

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Abstract

Recent studies have demonstrated that polygonal elements possess great potential in the study of nonlinear elastic materials under finite deformations. On the one hand, these elements are well suited to model complex microstructures (e.g. particulate microstructures and microstructures involving different length scales) and incorporating periodic boundary conditions. On the other hand, polygonal elements are found to be more tolerant to large localized deformations than the standard finite elements, and to produce more accurate results in bending and shear. With mixed formulations, lower order mixed polygonal elements are also shown to be numerically stable on Voronoi-type meshes without any additional stabilization treatment. However, polygonal elements generally suffer from persistent consistency errors under mesh refinement with the commonly used numerical integration schemes. As a result, non-convergent finite element results typically occur, which severely limit their applications. In this work, a general gradient correction scheme is adopted that restores the polynomial consistency by adding a minimal perturbation to the gradient of the displacement field. With the correction scheme, the recovery of optimal convergence for solutions of displacement-based and mixed formulations with both linear and quadratic displacement interpolants is confirmed by numerical studies of several boundary value problems in finite elasticity. In addition, for mixed polygonal elements, the various choices of the pressure field approximations are discussed, and their performance on stability and accuracy are numerically investigated. We present applications of those elements in physically-based examples including a study of filled elastomers with interphasial effect and a qualitative comparison with cavitation experiments for fiber reinforced elastomers.

Keywords: Gradient correction; Finite elasticity; Quadratic polygonal element; Mixed variational principle; Filled elastomers; Cavitation

1. Introduction

The finite element space for polygonal elements contains non-polynomial (e.g. rational) functions and thereby the existing quadrature schemes, typically designed for integration of polynomial functions, will lead to persistent

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consistency errors that do not vanish with mesh refinement [1,2]. As a direct consequence, the so-called patch test, which provides a measure of polynomial consistency of conforming discretizations, is not passed on general polygonal meshes, even in an asymptotic sense. Moreover, the persistence of the errors in turn renders the finite element method suboptimally convergent or even non-convergent under mesh refinement. In practice, using sufficiently large number of integration points can lower the consistency error. For linear polygonal elements in two dimensions (2D), a triangulation scheme with three integration points per triangle is shown to be sufficiently accurate for practical problems and mesh sizes [1,3]. However, for higher-order polygonal elements, for instance, quadratic elements that will be discussed in this paper, the number of integration points of such a scheme can become prohibitively large. This is also the case for polyhedral elements in three dimensions (3D). For example, maintaining optimal convergence rates with a linear polyhedral discretization for practical levels of mesh refinement may require several hundred integration points per element [2,4]. As the number of elements increase, the associated computational cost on a polyhedral discretization can become too expensive for practical applications.

Several attempts have been made in the literature to address this issue. For example, in the context of scalar diffusion problems, inspired by the virtual element method (VEM) [5–7], Talischi et al. [1] have proposed a polynomial projection approach to ensure polynomial consistency of the bilinear form, and thereby ensure satisfaction of the patch test and optimal convergence for both linear and quadratic polygonal elements. A similar approach has also been adopted by Manzini et al. [2] to solve Poisson problems on polyhedral meshes. However, those approaches require the existence of a bilinear form, and therefore extension to general nonlinear problems is non-trivial and is still an open question. Borrowing the idea of pseudo-derivatives in the meshless literature [8], Bishop has proposed an approach to correct the derivatives of the shape functions to enforce the linear consistency property on general polygonal and polyhedral meshes [9,10]. With the correction, the linear patch test is passed and optimal convergence is achieved. Although being applicable for general nonlinear cases, extension to higher order cases (e.g. quadratic polygonal finite elements) is not trivially implied.

More recently, Talischi et al. [4] have proposed a general gradient correction scheme that is applicable to both linear and nonlinear problems on polygonal and polyhedral elements with arbitrary orders. In essence, the scheme corrects the gradient field at the element level with a minimal perturbation such that the discrete divergence theorem is satisfied against polynomial functions of suitable order. With a minimum accuracy requirement of the numerical integration scheme, the correction has been previously shown to restore optimal convergence for linear diffusion and nonlinear Forchheimer flow problems [4]. In this work, we adopt the gradient correction scheme in two dimensional finite elasticity problems and apply it to linear and quadratic polygonal elements. As we will see, the gradient correction scheme renders both linear and quadratic polygonal elements optimally convergent.

To enable modeling of materials with a full range of compressibility, this work considers displacement-based as well as two-field mixed polygonal elements, the latter of which contains an additional discrete pressure field. For mixed finite elements, the numerical stability is a critical issue to ensure convergence and therefore has been subjected to extensive studies in the finite element literature. Generally, the stability condition is described by the well-known inf–sup condition [11–13], which, in essence, implies a balance between the discrete spaces for displacement field and pressure field [14]. Many of the classical mixed finite elements are known to be unstable (see, for instance, summaries in [15,16]). As a result, some post-processing procedures or stabilization methods are needed for those elements (see, for instance, [17–20]). In contrast, some recent contributions have suggested that linear mixed polygonal elements coupled with element-wise constant pressure field are numerically stable on Voronoi-type meshes in both linear and nonlinear problems if every node/vertex in the mesh is connected to at most three edges [14,21,3]. Furthermore, with the availability of higher order displacement interpolants (see, e.g. [22, 23]), various mixed approximations for higher order polygonal elements featuring more enriched pressure spaces are made possible. For example, for quadratic mixed elements, together with quadratic interpolation of the displacement field, the pressure field can be approximated by either discontinuous piecewise-linear or continuous linearly complete functions. However, their stability, convergence and accuracy are still open problems and have not been fully explored in the literature. In this paper, the performance of quadratic mixed polygonal elements with different choices of pressure approximation is presented and studied with thorough numerical assessment. As a direct observation, the quadratic mixed polygonal elements also appears to be stable with both discontinuous piecewise-linear and continuous linearly complete interpolations of the pressure field on Voronoi-type meshes for linear elasticity problems. Intuitively, this stability results from the larger displacement space for polygonal finite elements when compared with the classical triangular and quadrilateral elements.