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Space–time isogeometric analysis of parabolic evolution problems

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Abstract

We present and analyze a new stable space–time Isogeometric Analysis (IgA) method for the numerical solution of parabolic evolution equations in fixed and moving spatial computational domains. The discrete bilinear form is elliptic on the IgA space with respect to a discrete energy norm. This property together with a corresponding boundedness property, consistency and approximation results for the IgA spaces yields an a priori discretization error estimate with respect to the discrete norm. The theoretical results are confirmed by several numerical experiments with low- and high-order IgA spaces.

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Keywords: Parabolic initial–boundary value problems; Space–time isogeometric analysis; Fixed and moving spatial computational domains; A priori discretization error estimates

1. Introduction

Let us consider the parabolic initial–boundary value problem: find $u : \overline{Q} \to \mathbb{R}$ such that

$$\partial_t u - \Delta u = f \quad \text{in } Q, \qquad u = 0 \quad \text{on } \Sigma, \quad \text{and} \quad u = u_0 \quad \text{on } \Sigma_0, \tag{1.1}$$

as the typical model problem for a linear parabolic evolution equation posed in the space–time cylinder $\overline{Q} = \overline{\Omega} \times [0, T] = Q \cup \Sigma \cup \Sigma_0 \cup \Sigma_T$, where ∂_t denotes the partial time derivative, Δ is the Laplace operator, f is a given source function, u_0 are the given initial data, T is the final time, $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$, $\Sigma_0 := \Omega \times \{0\}$, $\Sigma_T := \Omega \times \{T\}$, and $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$ denotes the spatial computational domain with the boundary $\partial\Omega$. For the time being, we assume that the domain Ω is fixed, bounded and Lipschitz. In many practical, in particular, industrial applications, the domain Ω, that is also called physical domain, is usually generated by some CAD system, i.e., it can be represented by a single patch or multiple patches which are images of the parameter domain $(0, 1)^d$ by spline

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or NURBS maps. Parabolic initial–boundary value problems of the form (1.1) arise in many practical applications. For instance, heat conduction and diffusion processes but also evolution processes in life and social sciences can be modeled by (1.1) or by (1.1) with a more general elliptic part. Sometimes Eq. (1.1) is called heat conduction, diffusion, or potential equation. In electromagnetics, the z-component of the vector potential solves (1.1) in the case of a 2d eddy current problem.

The standard discretization methods in time and space are based on time-stepping methods combined with some spatial discretization technique like the Finite Element Method (FEM) [1,2]. The vertical method of lines discretizes first in time and then in space [1], whereas in the horizontal method of lines, also called Rothe's method, the discretization starts with respect to (wrt) the time variable [2]. The later method has some advantages wrt the development of adaptive techniques. However, in both approaches, the development of really efficient adaptive techniques suffers from the separation of the time and the space discretizations. Moreover, this separation is even more problematic in parallel computing. The curse of sequentiality of time affects the construction of efficient parallel methods and their implementation on massively parallel computers with several thousands or even hundreds of thousands of cores in a very bad way.

The simplest ideas for space–time solvers are based on time-parallel integration techniques for ordinary differential equations that have a long history, see [3] for a comprehensive presentation of this history. The most popular parallel time integration method is the parareal method that was introduced by Lions, Maday and Turinici in [4]. Time-parallel multigrid methods have also a long history. In 1984, Hackbusch proposed the so-called parabolic multigrid method that allows a simultaneous execution on a set of successive time steps [5]. Lubich and Ostermann [6] introduced parallel multigrid waveform relaxation methods for parabolic problems. A comprehensive presentation of these methods and a survey of the references until 1993 can be found in the monograph [7]. Vandewalle and Horton investigate the convergence behavior of these time-parallel multigrid methods by means of Fourier mode analysis [8]. Deshpande et al. provided a rigorous analysis of time domain parallelism [9]. Very recently, Gander and Neumüller have also used the Fourier analysis to construct perfectly scaling parallel space–time multigrid methods for solving initial value problems for ordinary differential equations [10] and initial–boundary value problems for parabolic PDEs [11]. In these two papers, the authors construct stable high-order dG discretizations in time slices. In [12], this technique is used to solve the arising linear system of a space–time dG discretization, which is also stable in the case of the decomposition of the space–time cylinder into 4d simplices (pentatopes) for 3d spatial computational domains. This idea opens great opportunities for flexible discretizations, adaptivity and the treatment of changing spatial domains in time [13,14,12,15]. We also refer to [16–25] where different space–time techniques have been developed. Babuška and Janik already developed h-p versions of the finite element method in space with p and h-p approximations in time for parabolic initial–boundary value problems in the papers [26] and [27], respectively. In [28], Schwab and Stevenson have recently developed and analyzed space–time adaptive wavelet methods for parabolic evolution problems, see also [29]. Similarly, Mollet proved uniform stability of an abstract Petrov–Galerkin discretizations of boundedly invertible operators and applied this result to space–time discretizations of linear parabolic problems [30]. Very recently Urban and Patera have proved error bounds for reduced basis approximation to linear parabolic problems [31], whereas Steinbach has investigated conforming space–time finite element approximations to parabolic problems [32]. Our approach uses special time-upwind test functions which are motivated by a space–time streamline diffusion method [33–36] and by a similar approach used in [37] and [38]. The increasing interest in highly time-parallel space–time methods is certainly connected with the fact that parallel computers have rapidly developed with respect to number of cores, computation speed, memory, availability etc., but also with the complexity of the problems the people want to solve. In particular, the optimization of products and processes on the basis of computer simulations of the underlying transient processes (PDE constraints) foster the development of space–time methods since the optimality system is basically nothing but a system of primal and adjoint PDEs which are coupled forward and backward in time, see, e.g., [39]. In fact, there are several papers on the use of various space time-methods for solving exciting engineering problems like fluid–structure interaction, aerodynamics problems and cardiac electro-mechanics see [40–45], and the references therein. These papers show the great potential of space–time methods for solving complex problems, in particular, problems with moving boundaries or interfaces. The use of B-splines and NURBS for both representing moving boundaries and approximating the solution of the PDE system we are looking for, enhances the accuracy and flexibility of the simulation considerably, see [40–44]. In combination with a full space–time adaptivity and parallelization in space and time, these methods can be very efficient on current and future computers with many thousand or even millions of cores. Whereas space–time IgA technologies are already used for solving engineering