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Three dimensional hierarchical mixed finite element approximations with enhanced primal variable accuracy

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Abstract

There are different possibilities of choosing balanced pairs of approximation spaces for dual (flux) and primal (pressure) variables to be used in discrete versions of the mixed finite element method for elliptic problems arising in fluid simulations. Three cases shall be studied and compared for discretized three dimensional formulations based on tetrahedral, hexahedral and prismatic meshes. The principle guiding the constructions is the property that the divergence of the dual space and the primal approximation space should coincide, while keeping the same order of accuracy for the flux variable and varying the accuracy order of the primal variable. There is the classic case of BDM_k spaces based on tetrahedral meshes and polynomials of total degree k for the dual variable, and $k-1$ for the primal variable, showing stable simulations with optimal convergence rates of orders $k+1$ and k, respectively. Another case is related to RT_k and $BDFM_{k+1}$ spaces for hexahedral and tetrahedral meshes, respectively, but holding for prismatic elements as well. It gives identical approximation order $k+1$ for both primal and dual variables, an improvement in accuracy obtained by increasing the degree of primal functions to k, and by enriching the dual space with some properly chosen internal shape functions of degree $k+1$, while keeping degree k for the border fluxes. A new type of approximation is proposed by further incrementing the order of some internal flux functions to $k+2$, and matching primal functions to $k+1$ (higher than the border fluxes of degree k). Thus, higher convergence rate of order $k+2$ is obtained for the primal variable. Using static condensation, the global condensed system to be solved in all the cases has same dimension (and structure), which is proportional to the space dimension of the border fluxes for each element geometry. Illustrating results comparing the three different space configurations are presented for simulations based on hierarchical high order shape functions for $\mathbf{H}$(div)-conforming spaces,

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which are specially constructed for affine tetrahedral, hexahedral and prismatic meshes. Expected convergence rates are obtained for the flux, pressure and divergence variables.

Keywords: Finite elements; **H**(div) spaces; Mixed formulation; Approximation space configurations; Convergence rates; 3D meshes

1. Introduction

Mixed finite element methods have the ability to provide accurate and locally conservative fluxes, an advantage over standard H^1-finite element discretizations [1]. They are based on simultaneous approximations of the primal (pressure) and dual (flux) variables, involving two kinds of approximation spaces. In addition to **H**(div)-conforming approximation spaces for the dual variable, with continuous normal components over element interfaces, the primal variable is usually represented in discontinuous finite element spaces.

Since the pioneering work by Raviart and Thomas [2] in 1977, different constructions of **H**(div) (or **H**(curl)) approximation spaces have been proposed in [3–7]. Recently, several other papers have appeared in the literature due to increasing interest on this subject. In some contexts the vector basis functions are constructed directly on the physical element, but when it comes to solving practical problems in complex domains, shape functions are defined on the master element and then they are transformed to the elements of the partition using Piola transformations. Constructions of hierarchical high order spaces in [8–13] are based on the properties of the De Rham complex, which are essential in the stability proof. There is also the methodology based on finite element exterior calculus [14,15] that can be used for the construction of stable discretizations of a variety of problems in a unified framework, which is used in [16] for the implementation of some classic simplicial space configurations in the FEniCS software. In [17] and [18] innovative families of spaces are presented for quadrilateral meshes in order to improve convergence rates in flux divergence, a common deficiency for general elements of other constructions.

The main purpose of this article is to analyse and compare the performance of different ways of choosing balanced pairs of approximation spaces $(\mathbf{V}_h, U_h)$, for dual and for primal variables, based on tetrahedral, hexahedral and prismatic meshes, to be used in discrete versions of the mixed finite element method for three dimensional elliptic problems. The methods share the following basic characteristics:

1. The flux approximation spaces $\mathbf{V}_h$ shall be spanned by a hierarchy of vectorial shape functions, which are organized into two classes: the shape functions of interior type, with vanishing normal components over all element faces, and the shape functions associated to the element faces.
2. In all the cases, as in the de Rham complex, the aim is to verify the discrete exact sequence property
$$\nabla \cdot \mathbf{V}_h = U_h, \tag{1}$$
in order to obtain stable results with optimal L^2-error convergence orders, which are dictated by the degree of the complete set of polynomials used to form the corresponding approximation spaces.

For the classic Brezzi–Douglas–Marini (BDM_k) spaces applied to triangular elements [3], and their generalization to tetrahedral geometries by Nédélec [5], stable approximations can be obtained with spaces of type $\mathbf{P}_k\ P_{k-1}$, which are based on complete scalar polynomials of total degree $k-1$ for approximations of the primal variable, and of total degree k for the components of **H**(div) approximations of the dual variable. In these cases, property (1) holds, resulting in stable simulations with optimal convergence rates, in the L^2-norm, of orders $k+1$ and k for dual and primal variables, respectively.

For hexahedral, and prismatic 3D geometries, and also for quadrilateral 2D meshes, approximation spaces of type $\mathbf{P}_k\ P_{k-1}$ cannot verify property (1), and thus they are not consistent. However, approximation spaces of type $\mathbf{P}_k^*\ P_k$ are possible to construct for these geometries, and also for triangles, quadrilaterals, and tetrahedra as well, with optimal convergence rates of order $k+1$ for both dual and primal variables. The principle for the construction of $\mathbf{P}_k^*$ type of flux approximations is to enrich the dual space $\mathbf{P}_k$ with some properly chosen internal shape functions of degree $k+1$, while keeping degree k for the border fluxes. This type of space configuration was first introduced by Raviart–Thomas (RT_k) for quadrilateral geometries, and its generalization to three dimension in [4]. This idea has also being used for the construction of the classic Brezzi–Douglas–Fortin–Marini ($BDFM_{k+1}$) elements for triangular