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Robust residual-based a posteriori Arnold–Winther mixed finite element analysis in elasticity[☆]

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Dedicated to Professor Ivo Babuška

Abstract

This paper presents a residual-based a posteriori error estimator for the Arnold–Winther mixed finite element that utilises a post-processing for the skew-symmetric part of the strain. Numerical experiments verify the proven reliability and efficiency for suitable approximation of the skew-symmetric deformation gradient. Numerical evidence supports that the L^2-stress error estimator is robust in the Poisson ratio and allows stable error control even in the incompressible limit.

Keywords: Finite element method; Arnold–Winther; A posteriori error analysis; Robustness; Elasticity

1. Introduction

The problem in linear elasticity considers the connected reference configuration of the elastic body $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ with closed and connected Γ_D of positive surface measure and $\Gamma_N = \partial\Omega \setminus \Gamma_D$ for applied tractions. Given a volume force $f : \Omega \to \mathbb{R}^2$, a displacement $u_D : \Gamma_D \to \mathbb{R}^2$, and a traction $g : \Gamma_N \to \mathbb{R}^2$, find a displacement $u : \Omega \to \mathbb{R}^2$ and a stress tensor $\sigma : \Omega \to \mathbb{S} := \{\tau \in \mathbb{R}^{2\times 2} : \tau = \tau^T\}$ such that

$$\begin{aligned} -\operatorname{div}\sigma = f, \quad \sigma = \mathbb{C}\varepsilon(u) \quad \text{in } \Omega,\\ u = u_D \quad \text{on } \Gamma_D, \quad \sigma\nu = g \quad \text{on } \Gamma_N. \end{aligned} \tag{1.1}$$

Throughout this paper, $\mathbb{C}$ denotes the bounded and positive definite fourth-order elasticity tensor for isotropic linear elasticity. The symmetric mixed finite element method is a very popular choice for a robust stress approximation; cf. [1–6] for details and related references.

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The a posteriori error analysis for the Arnold–Winther finite element method may follow the ideas of [7–9] to derive a stress error control

$$\|\sigma - \sigma_{AW}\|^2_{\mathbb{C}^{-1}} \le \min_{v\in V} \|\mathbb{C}^{-1}\sigma_{AW} - \varepsilon(u_D + v)\|^2_{\mathbb{C}} + C_1 \mathrm{osc}^2(f, \mathcal{T}) + C_2 \mathrm{osc}^2(g, \mathcal{E}(\Gamma_N))$$

for the stress error $\sigma - \sigma_{AW}$ even with a rather explicit estimate of the constant in front of the oscillations and the (unwritten) multiplicative constant 1 in front of the first term that measures the quality of the approximation $\mathbb{C}^{-1}\sigma_{AW}$ of symmetric gradients $\varepsilon(v) := (Dv + D^T v)/2$ for $v \in V$. The space V consists of all square-integrable displacements with homogeneous boundary conditions along Γ_D and with a square-integrable functional matrix Dv.

A severe additional difficulty of this approximation is that only the symmetric part is approximated and not the full gradient Dv so that [9] cannot be applied for a residual-based a posteriori error estimation of the aforementioned first term. Other mixed finite element schemes like PEERS in [7] involve some additional variable to approximate the asymmetric part of the gradient. This paper presents an explicit error estimate which involves an arbitrary asymmetric approximation γ_h and provides an abstract a posteriori error control of the residual type, which is useful for adaptive mesh-refining algorithms,

$$\eta_\ell^2 = \mathrm{osc}^2(f, \mathcal{T}) + \mathrm{osc}^2(g, \mathcal{E}(\Gamma_N)) + \sum_{T\in\mathcal{T}} h_T^2 \|\operatorname{curl}(\mathbb{C}^{-1}\sigma_{AW} + \gamma_h)\|^2_{L^2(T)}$$
$$+ \sum_{E\in\mathcal{E}(\Omega)} h_E \|[\mathbb{C}^{-1}\sigma_{AW} + \gamma_h]_E \tau_E\|^2_{L^2(E)} + \sum_{E\in\mathcal{E}(\Gamma_D)} h_E \|(\mathbb{C}^{-1}\sigma_{AW} + \gamma_h - Du_D)\tau\|^2_{L^2(E)}.$$

(The details on the standard notation can be found below for computable volume contributions on a triangle T of diameter h_T and various jumps across an edge E of length h_E.) For any (piecewise smooth) choice of γ_h, this a posteriori error estimator is reliable in the sense that

$$\|\sigma - \sigma_{\mathrm{AW}}\|_{\mathbb{C}^{-1}} \le C_{\mathrm{rel}} \eta_\ell \tag{1.2}$$

with some λ-independent constant $C_{\mathrm{rel}} \approx 1$. One opportunity to ensure efficiency is a global minimisation over all piecewise polynomial γ_h of the error estimator η_ℓ. The bubble function technique shows that the particular choice of γ_h enters the efficiency estimates with some λ-independent constant $C_{\mathrm{eff}} \approx 1$,

$$\eta_\ell \le C_{\mathrm{eff}} \left(\|\sigma - \sigma_{\mathrm{AW}}\|_{\mathbb{C}^{-1}} + \|\mathrm{skew}(Du) - \gamma_h\|_{L^2(\Omega)}\right). \tag{1.3}$$

Hence, one efficient choice for γ_h is to choose it as a sufficiently accurate polynomial approximation of the asymmetric gradient $\mathrm{skew}(Du) := (Du - D^T u)/2$. Since a global approximation or even minimisation may be too costly, this paper proposes to apply a post-processing step to compute such a sufficiently accurate approximation $\gamma_h = \mathrm{skew}(Du^*_{AW})$ for the post-processed displacement u^*_{AW} in the spirit of Stenberg [10]. The approximation $\gamma_h = \mathrm{skew}(Du^*_{AW})$ is proven to be robust in the Poisson ratio $\nu \to 1/2$ for sufficiently smooth functions. For domains with re-entrant corners or incompatible boundary conditions, numerical experiments confirm that the proposed computation of γ_h leads empirically to reliable and efficient a posteriori error control independent of the Poisson ratio $\nu \to 1/2$.

The remaining parts of this paper are organised as follows. In Section 2 the notation, the weak formulation of (1.1) and the Arnold–Winther finite element space [1] are defined. Section 3 derives the a posteriori error analysis for the residual-based a posteriori error estimator and proves reliability and efficiency. Section 4 outlines a post-processing of the displacement that leads to an approximation γ_h of the asymmetric gradient. Section 5 presents numerical results of four benchmark problems that verify reliability and efficiency of the residual-based a posteriori error estimator in combination with the post-processing and illustrates its robustness for Poisson ratio $\nu \to 1/2$.

The main parts of this research are restricted to 2D because the Argyris finite element method is employed to allow for a quasi-interpolation in the Arnold–Winther finite element functions.

2. Preliminaries

For $v = (v_1, v_2) \in \mathbb{R}^2$ and $\tau = (\tau_{jk})_{j,k=1,2} \in \mathbb{R}^{2\times 2}$, set

$$\mathrm{Curl}(v) := \begin{pmatrix} \partial v_1/\partial y & -\partial v_1/\partial x \\ \partial v_2/\partial y & -\partial v_2/\partial x \end{pmatrix}, \quad \operatorname{curl} \tau := \begin{pmatrix} \partial \tau_{12}/\partial x - \partial \tau_{11}/\partial y \\ \partial \tau_{22}/\partial x - \partial \tau_{21}/\partial y \end{pmatrix}, \quad \operatorname{div} \tau := \begin{pmatrix} \partial \tau_{11}/\partial x + \partial \tau_{12}/\partial y \\ \partial \tau_{21}/\partial x + \partial \tau_{22}/\partial y \end{pmatrix}.$$