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High order unfitted finite element methods on level set domains using isoparametric mappings

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Abstract

We introduce a new class of unfitted finite element methods with high order accurate numerical integration over curved surfaces and volumes which are only implicitly defined by level set functions. An unfitted finite element method which is suitable for the case of piecewise planar interfaces is combined with a parametric mapping of the underlying mesh resulting in an isoparametric unfitted finite element method. The parametric mapping is constructed in a way such that the quality of the piecewise planar interface reconstruction is significantly improved allowing for high order accurate computations of (unfitted) domain and surface integrals. We present the method, discuss implementational aspects and present numerical examples which demonstrate the quality and potential of this method.

Keywords: Numerical integration; Level set; Unfitted finite element method; Isoparametric finite element method; High order methods; Interface problems

1. Introduction

1.1. Motivation

In the recent years *unfitted finite element methods* have drawn more and more attention. The possibility to handle complex geometries without the need for complex and time consuming mesh generation is very appealing. The methodology of unfitted finite element methods, i.e. methods which are able to cope with interfaces or boundaries which are not aligned with the grid, has been investigated for different problems. For boundary value problems with non-aligned boundaries methods such as penalty methods [1,2], the fictitious domain method [3,4], the immersed boundary method [5], and other unfitted finite element methods [6] have been developed. For *unfitted* interface problems extended finite element methods (XFEM) have been developed in (among others) [7–11]. Also partial differential equations on surfaces which are not meshed have been considered using the trace finite element method (TraceFEM) [12].

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In the community of unfitted finite element methods mostly piecewise linear discretizations are considered. One major issue in the design and realization of high order methods is the problem of numerical integration on domains which are only prescribed implicitly, for instance as the zero level of a scalar function, the so-called level set function. The use of standard integration rules (which ignore the cut position on cut elements) is not a good option as the integrand does not provide the necessary smoothness.

The purpose of this paper is the presentation of a new approach which allows for high order numerical integration on domains prescribed by level set functions. The approach is new, robust and fairly simple to implement. The method is geometry-based and can be applied to unfitted interface or boundary value problems as well as to partial differential equations on surfaces.

1.2. Literature

We briefly discuss the state of the art in the literature to put the new approach into context. One approach that is often used in order to realize numerical integration on implicit domains consists of two steps: the approximation of the interface with a piecewise planar interface and a tessellation algorithm to divide the subdomains of a cut element into simple geometries, on which standard quadrature rules can be applied. A famous example for quadrilaterals and cubes is the marching cube algorithm [13]. For simplices a detailed discussion of this approach can be found in (among others) [14, Chapter 5], [15] for triangles and tetrahedra and in [16,17] also for 4-prisms and pentatopes (4-simplices). Many simulation codes which apply unfitted finite element discretizations, e.g. [18–22] make use of such strategies. In order to increase the accuracy of this integration one often combines this approach with subdivisions and adaptivity. Especially on octree-based meshes this can be done very efficiently [23]. However, this tessellation approach is only second order accurate (w.r.t. the finest subtriangulation) which complicates the realization of high order methods.

One approach to solve this problem is the tailoring of quadrature points and weights which provide high order accurate integration rules for implicit domains. Such a construction of points and weights is based on the idea of fitting integral moments, cf. [24,25]. Although this results in accurate integration rules it has two shortcomings. First, the computation of quadrature rules is fairly involved. This aspect is typically outwayed by the resulting high accuracy. Secondly, the major problem, quadrature weights are in general not positive. This can lead to stability problems as positivity of mass or stiffness matrices in finite element formulations can no longer be guaranteed.

For special cases satisfactory answers to the question of high order numerical integration strategies have been found which allow for an implementation of high order unfitted finite element methods. We mention a few interesting approaches. For quadrilaterals and hexahedra in [26] a numerical integration algorithm is presented which can achieve arbitrary high order accuracy and guarantee positivity of integration weights at the same time. The approach is based on the idea of interpreting the interface as a piecewise graph over hyperplanes. In [27] an unfitted boundary value problem is considered. Instead of aiming at a high order accurate geometrical approximation of the boundary a correction in the imposition of boundary values is applied in order to recover a high order method. For the discretization of partial differential equations on surfaces, in [28] a *parametric mapping of the interface* from a piecewise planar representation to a high order representation based on quasi-normal fields is applied. Although the approach cannot be carried over straight-forwardly to the case where also the integration on sub-domains is important, that approach has important similarities to the approach presented in this paper.

In the literature of extended finite element methods (XFEM), there exist other approaches which are based on a tessellation approach and aim at improving the resulting piecewise planar approximations by means of a *parametric mapping of the sub-triangulation* [29–31]. These approaches are typically technically involved, especially in more than two dimensions. Moreover, ensuring robustness of these approaches is difficult.

The approach presented in this paper is similar to the approaches in [28–30] in that it is also based on a piecewise planar interface which is significantly improved using a parametric mapping. The important difference is, that we consider a *parametric mapping of the underlying mesh* rather than the sub-triangulation or the interface. According to the mapping of the mesh the use of isoparametric finite element is natural.

At this point, we would also like to mention the paper [32] where the construction of a mesh deformation, which is also used in combination with isoparametric finite elements, is specifically designed to align a given mesh to a given interface position. The goal in that paper is to obtain a mesh that is *conforming* to a given interface without changing the mesh topology while keeping the quality of the mesh. Our approach is in a similar virtue.