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An extended range of stable-symmetric-conservative Flux Reconstruction correction functions

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Abstract

The Flux Reconstruction (FR) approach offers an efficient route to achieving high-order accuracy on unstructured grids. Additionally, FR offers a flexible framework for defining a range of numerical schemes in terms of so-called FR correction functions. Recently, a one-parameter family of FR correction functions were identified that lead to stable schemes for 1D linear advection problems. In this study we develop a procedure for identifying an extended range of stable, symmetric, and conservative FR correction functions. The procedure is applied to identify ranges of such correction functions for various orders of accuracy. Numerical experiments are undertaken, and the results found to be in agreement with the theoretical findings.

Keywords: High-order methods; Flux Reconstruction; Discontinuous Galerkin method; Spectral difference method; Energy stability

1. Introduction

High-order methods for computational aerodynamics on unstructured grids offer the promise of increased accuracy at reduced cost, within the vicinity of complex engineering geometries. As such they have garnered continued interest over the past decades. However, to-date, their 'real-world' adoption in both industry and academia remains limited [1]. In 2007 Huynh proposed the Flux Reconstruction (FR) approach to high-order methods [2]. Based on a differential form of the governing system, it is hoped FR (also referred to as Lifting Collocation Penalty [3] or Correction Procedure via Reconstruction [4]) will facilitate adoption of high-order methods amongst a wider community of fluid dynamicists.

Various properties of FR schemes, including their dispersion and dissipation characteristics [5,6], their associated Courant–Friedrichs–Lewy (CFL) limit [2,5], and their fundamental stability [7], are all determined in full or in part by the form of their associated FR correction functions. These correction functions act to lift inter-element flux jumps from the boundary into the interior of each element. Building on the work of Huynh [2] and Jameson [8],

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Vincent, Castonguay and Jameson recently identified a one-parameter family of correction functions that lead to stable FR schemes for 1D linear advection problems [7]. Identification of these correction functions, henceforth referred to as Vincent–Castonguay–Jameson–Huynh (VCJH) correction functions, provided significant insight into stability properties various FR schemes. However, further work is required in order to determine a full specification of the necessary and sufficient conditions that should be imposed on correction functions in order to guarantee stability.

In this study we develop a procedure for identifying an extended range of stable, symmetric, and conservative FR correction functions. The procedure is applied to identify ranges of such correction functions for various orders or accuracy. In all cases the original one-parameter VCJH correction functions are found to be a sub-set of the extended ranges. Numerical experiments are undertaken in order to verify the theoretical findings.

2. Flux reconstruction

2.1. Overview

FR schemes are similar to nodal DG schemes, which are arguably the most popular type of unstructured high-order method (at least in the field of computational aerodynamics). Like nodal DG schemes, FR schemes utilise a high-order (nodal) polynomial basis to approximate the solution within each element of the computational domain, and like nodal DG schemes, FR schemes do not explicitly enforce inter-element solution continuity. However, unlike nodal DG schemes, FR methods are based solely on the governing system in a differential form. A description of the FR approach in 1D is presented below. For further information see the original paper of Huynh [2].

2.2. Preliminaries

Consider solving the following 1D scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{2.1}$$

within an arbitrary domain $\boldsymbol{\Omega}$, where x is a spatial coordinate, t is time, $u = u(x, t)$ is a conserved scalar quantity and $f = f(u)$ is the flux of u in the x direction. Additionally, consider partitioning $\boldsymbol{\Omega}$ into N distinct elements, each denoted $\boldsymbol{\Omega}_n = \{x | x_n < x < x_{n+1}\}$, such that

$$\boldsymbol{\Omega} = \bigcup_{n=0}^{N-1} \boldsymbol{\Omega}_n, \qquad \bigcap_{n=0}^{N-1} \boldsymbol{\Omega}_n = \emptyset. \tag{2.2}$$

The FR approach requires u is approximated in each $\boldsymbol{\Omega}_n$ by a function $u_n^\delta = u_n^\delta(x, t)$, which is a polynomial of degree k within $\boldsymbol{\Omega}_n$, and identically zero elsewhere. Additionally, the FR approach requires f is approximated in each $\boldsymbol{\Omega}_n$ by a function $f_n^\delta = f_n^\delta(x, t)$, which is a polynomial of degree $k + 1$ within $\boldsymbol{\Omega}_n$, and identically zero elsewhere. Consequently, when employing the FR approach, a total approximate solution $u^\delta = u^\delta(x, t)$ and a total approximate flux $f^\delta = f^\delta(x, t)$ can be defined within $\boldsymbol{\Omega}$ as

$$u^\delta = \sum_{n=0}^{N-1} u_n^\delta \approx u, \qquad f^\delta = \sum_{n=0}^{N-1} f_n^\delta \approx f, \tag{2.3}$$

where no level of inter-element continuity in u^δ is explicitly enforced. However, f^δ is required to be C_0 continuous at element interfaces.

Note the requirement that each f_n^δ is one degree higher than each u_n^δ, which consequently ensures the divergence of f_n^δ is of the same degree as u_n^δ within $\boldsymbol{\Omega}_n$.

2.3. Implementation

From an implementation perspective, it is advantageous to transform each $\boldsymbol{\Omega}_n$ to a standard element $\boldsymbol{\Omega}_S = \{\hat{x} | -1 \le \hat{x} \le 1\}$ via the mapping

$$\hat{x} = \Gamma_n(x) = 2\left(\frac{x - x_n}{x_{n+1} - x_n}\right) - 1, \tag{2.4}$$