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On some time marching schemes for the stabilized finite element approximation of the mixed wave equation

Hector Espinoza[a,b], Ramon Codina[a,*], Santiago Badia[a,c]

[a] Universitat Politècnica de Catalunya, Jordi Girona 1-3, Edifici C1, E-08034 Barcelona, Spain
[b] Escuela Superior Politécnica del Litoral ESPOL, Km 30.5 vía Perimetral, EC090902, Guayaquil, Ecuador
[c] Centre Internacional de Mètodes Numèrics en Enginyeria, Parc Mediterrani de la Tecnologia, Esteve Terradas 5, E-08860 Castelldefels, Spain

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Abstract

In this paper we analyze time marching schemes for the wave equation in mixed form. The problem is discretized in space using stabilized finite elements. On the one hand, stability and convergence analyses of the fully discrete numerical schemes are presented using different time integration schemes and appropriate functional settings. On the other hand, we use Fourier techniques (also known as von Neumann analysis) in order to analyze stability, dispersion and dissipation. Numerical convergence tests are presented for various time integration schemes, polynomial interpolations (for the spatial discretization), stabilization methods, and variational forms. To analyze the behavior of the different schemes considered, a 1D wave propagation problem is solved.

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Keywords: Time marching schemes; Dispersion; Dissipation; von Neumann analysis; Fourier analysis; Mixed wave equation

1. Introduction

Finite difference time marching schemes are mostly used for the time integration of evolution problems because of their efficiency and ease of implementation. In the case of partial differential equations (PDEs) in space and time, even if a given finite difference scheme has some general properties regarding stability and accuracy, the precise behavior of the scheme needs to be analyzed together with the spatial discretization employed. In this paper we aim at analyzing classical first and second order schemes for the hyperbolic wave equation, with the particularity that we write it in mixed form and discretize in space using stabilized finite element (FE) methods.

For the analysis of time discretization schemes for wave propagation problems it is customary to split the total discretization error into two parts: dispersion error and dissipation error. Dispersion is the dependency of the phase

* Corresponding author. Tel.: +34 934016486.
E-mail addresses: hector.espinoza@upc.edu (H. Espinoza), ramon.codina@upc.edu (R. Codina), sbadia@cimne.upc.edu (S. Badia).
URLs: https://upc.edu/espinoza (H. Espinoza), http://codina.rmee.upc.edu/ (R. Codina), http://badia.rmee.upc.edu/ (S. Badia).

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velocity on the frequency and the dispersion error is the deviation of the phase velocity with respect to the expected one for a given frequency. On the other hand, dissipation error is the decrease in amplitude with respect to the expected one. In general, dispersion and dissipation errors are higher for poorly resolved frequencies, which occurs mainly for high frequencies. The wave equation we aim to analyze is non-dispersive and non-dissipative. Therefore, it would be desirable to have non-dispersive and non-dissipative discretization schemes. This sometimes cannot be achieved, thus one just aims to have low-dispersion low-dissipation schemes [1,2]. Dispersion and/or dissipation of numerical schemes can be evaluated using Fourier techniques (see [3–5]), energy methods (see [6]), or modified equation analysis (see [4,7]). Fourier analysis can be carried out for semi-discretizations or full discretizations. Dispersion/dissipation analysis methods have been used to optimize numerical schemes [3,7]. Other properties of the continuous wave equation, such as the preservation of symplecticity, some invariants or some symmetries are even more difficult to inherit for discrete schemes. This is the motivation of the so-called geometric numerical integrators, which we will not consider in this paper (see [8], for example).

Contrary to the irreducible hyperbolic wave equation, which is of second order in space and time, the mixed wave equation is of first order in space and time. We will consider the case of a single scalar unknown, which in the mixed format unfolds into two unknowns, namely, this scalar field and a vector unknown. With regard to the space approximation, the Galerkin FE discretization of this mixed wave equation requires to satisfy a compatibility condition between the spaces of the two unknowns (scalar and vector), i.e., to use so-called inf–sup compatible interpolations. Alternatively, we can consider stabilized FE methods, which provide much more flexibility when choosing the interpolation spaces [9–11]. In particular, we can consider equal interpolation for the unknowns. Stability and convergence of the stabilized FE spatial semi-discretization of the mixed wave equation has been presented in [10,11]. Stability and convergence of fully discrete schemes have also been analyzed for the convection–diffusion equation and the Stokes equations in [12–14] using spatial and temporal approximations related to those used in the present work. Note that the use of stabilized FE methods for general first order hyperbolic equations is an old topic, which can be traced back to [15] if only the advective terms are considered. However, we exploit here the particular structure of the wave equation and the functional settings that can be associated to it; this allows us to go much further in the analysis.

In this work, we analyze the stability and convergence properties for the mixed wave equation, after time semi-discretization, space semi-discretization, and full discretization, and perform their Fourier analyses. For the time discretization, we consider backward Euler (BE), Crank–Nicolson[1] (CN) and the second order backward differentiation formula (BDF2). We will see how a symplectic time integrator (CN) compares to non-symplectic time integrators (BE and BDF2). Dispersion and dissipation of discretization methods will be evaluated through numerical experiments. This consists in solving a given problem and evaluating the solution obtained [17,18,3]. We will solve a 1D wave propagation problem to show qualitatively dispersion and dissipation of the proposed numerical schemes.

The organization of the paper is as follows. In Section 2 we present the problem statement and its space–time discretization. In Section 3 we present stability and convergence results of the fully discrete problem obtained using variational techniques. We provide results for all the methods considered, even though we only present one sample of the proofs of these results. In Section 4 we present a complete Fourier analysis for the 1D wave equation in mixed form, from which precise information on the behavior of the different schemes can be drawn. Numerical results are presented in Section 5 and, finally, in Section 6 the conclusions of the work are summarized. This paper is a continuation of our work on the approximation of the mixed form of the wave equation presented in [10,11]. Frequent reference is made to these two papers, to which the reader is addressed for details.

2. Problem statement and numerical approximation

2.1. Initial and boundary value problem

The problem we consider is an initial and boundary value problem posed in a time interval $\Upsilon := (0, T)$ and in a spatial domain $\Omega \subset \mathbb{R}^d$, $(d = 1,\ 2 \text{ or } 3)$. Let $t \in \Upsilon$ be a given time instant in the temporal domain and $\boldsymbol{x} \in \Omega$ a given

[1] The original CN discretization scheme was devised to solve numerically PDEs of heat-conduction type; it is a space–time discretization based on finite differences. Sometimes CN is used to refer to the implicit midpoint method or the (implicit) trapezoidal rule and there is no agreement in the literature [16]. We have to mention that for linear operators (which is the case of the mixed wave equation) the trapezoidal rule and the implicit midpoint method are equivalent.