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Characteristic cut finite element methods for convection–diffusion problems on time dependent surfaces

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Abstract

We develop a finite element method for convection–diffusion problems on a given time dependent surface, for instance modeling the evolution of a surfactant. The method is based on a characteristic-Galerkin formulation combined with a piecewise linear cut finite element method in space. The cut finite element method is constructed by embedding the surface in a background grid and then using the restriction to the surface of a finite element space defined on the background grid. The surface is allowed to cut through the background grid in an arbitrary fashion. To ensure stability and well posedness of the resulting algebraic systems of equations, independent of the position of the surface in the background grid, we add a consistent stabilization term. We prove error estimates and present confirming numerical results.

Keywords: Cut finite element method; Surfactants; PDEs on surfaces; Characteristic Galerkin method

1. Introduction

Surfactants are important because of their ability to reduce the surface tension and are for example used in detergents, oil recovery, and in treatment of lung diseases [1]. In order to perform realistic simulations, development of numerical methods that accurately and efficiently solve the partial differential equation (PDE) for the evolution of the concentration of surfactants on moving and deforming interfaces is necessary. The effort put into this research area has therefore been extensive.

In order to solve for the concentration of surfactants on a moving interface there is a need to represent the interface. Techniques to represent the interface can be roughly divided into two classes: (1) explicit representation, e.g. by marker particles, and (2) implicit representation, e.g. by the level set of a higher dimensional function. In general, strategies for solving quantities on evolving surfaces are developed on the basis of the interface representation technique. Therefore, existing methods are usually tightly coupled to the interface representation, and several methods have

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been proposed both based on explicit [2–4] and implicit [5–10] representations. Implicit representation techniques have the benefit that it is usually not more complicated to solve the equations in three space dimensions compared to two space dimensions. However, the quantity on the interface is often extended to a region embedding the interface, and instead of a surface Partial Differential Equation (PDE), a PDE on a higher dimensional domain must be solved and precautions have to be taken in order to ensure good conservation of the surfactant mass.

A common strategy in the finite element framework is to let the mesh conform to the time-dependent interface see *e.g.*, [11,12]. This technique requires remeshing as an interface evolves with time and leads to significant complications when topological changes such as drop-breakup or coalescence occur, especially in three space dimensions. Recently, a different strategy was proposed in [13] for solving the Laplace–Beltrami operator, where the interface is embedded into a three dimensional background grid and the finite element space on the interface is the restriction of a finite element space defined on the background grid. We call this type of methods cut finite element methods (CutFEM) since the surface cuts through the background grid without restrictions. The main advantage of this approach is that it is very convenient when solving problems that involve coupled physical phenomena in the bulk domain and on surfaces or interfaces since the same finite element spaces can be used for all problems. Furthermore, it can be used with both explicit and implicit surface representations. In case of an implicit surface representation, the same background grid may be used both for the surface representation and for the problem on the surface. Recently a cut finite element method for solving PDEs on evolving surfaces based on a space–time approach was proposed and analyzed in [14], see also [15]. The error estimates presented in [14], however, use weaker norms than the L^2 norm and it appears optimal L^2 estimates have not yet been presented for a space–time approach. The method also needs to be stabilized when the problem is convection dominated. In [16] a space–time formulation for coupled bulk-surface problems modeling soluble surfactants was proposed.

A general difficulty of methods involving cut elements is that the stiffness matrix may become arbitrarily ill conditioned depending on the position of the surface in the background mesh. In the case of the Laplace–Beltrami operator the ill conditioning can be handled using a scaling, see [17]. Another alternative, which may be more suitable for more complex problems, is to add stabilization terms, see [18].

In this paper we extend the cut finite element method to a time dependent convection–diffusion equation on a given evolving surface governing the evolution of the surfactant concentration on a moving surface [19,20]. The strategy is to combine the characteristic Galerkin scheme [21] with a stabilized version of the cut finite element method. The stabilization is consistent and provides control over the jump in the normal gradient on the faces of the background grid. In the characteristic Galerkin scheme advection is treated by tracking information backward along the characteristics. Typically the CFL restriction is relaxed and also there is no need to stabilize the finite element method in case the problem is convection dominated. The method we propose is stable and yields a linear algebraic system of equations with bounded condition number independently of how the interface cuts the background grid. We use linear finite elements in space and a second order characteristic Galerkin scheme, which results in first order convergence in the L^2 norm when the time step k is proportional to the spatial mesh size h. We present a complete a priori error analysis for the method that takes discretization in space and time as well as the discrete approximation of the surface into account. The analysis utilizes the additional stability provided by the consistent stabilization term. Furthermore, the total mass of surfactant may be accurately conserved using a Lagrange multiplier. The proposed finite element method is straightforward to implement and works both with explicit and implicit interface representation techniques.

The remainder of the paper is organized as follows. In Section 2 we formulate the convection–diffusion equation. In Section 3 we present the finite element method. We summarize some preliminary results in Section 4 and derive error estimates in Section 5. In Section 6 we show numerical examples in two space dimensions. Finally, in Section 7 we summarize our results.

2. The continuous problem

2.1. The surface

Let $I = [0, T]$ be a time interval and let $\Sigma(t)$ be a smooth closed $d - 1$ dimensional surface embedded in $\mathbf{R}^d$, with $d = 2$ or 3, which evolves smoothly in time without any self intersection. We let $\rho(t) : \mathbf{R}^d \to \mathbf{R}$ be the signed distance function defined by $\rho(t, \boldsymbol{x}) = \min_{\boldsymbol{y}\in\Sigma(t)} |\boldsymbol{x} - \boldsymbol{y}|$ on the outside of $\Sigma(t)$ and $\rho(t, \boldsymbol{x}) = -\min_{\boldsymbol{y}\in\Sigma(t)} |\boldsymbol{x} - \boldsymbol{y}|$