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# Non-intrusive reduced order modelling of the Navier–Stokes equations

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## Highlights

- This is the first work to apply two non-intrusive ROMs for solving the N–S equations.
- A second order Taylors series method for calculating the POD coefficients.
- A Smolyak sparse grid collocation method for calculating the POD coefficients.
- Implementation of the non-intrusive ROMs does not require modifications to a system code.
- Ability of non-intrusive ROMs to capture the highly nonlinear fluid dynamics.

## Abstract

This article presents two new non-intrusive reduced order models based upon proper orthogonal decomposition (POD) for solving the Navier–Stokes equations. The novelty of these methods resides in how the reduced order models are formed, that is, how the coefficients of the POD expansions are calculated. Rather than taking a standard approach of projecting the underlying equations onto the reduced space through a Galerkin projection, here two different techniques are employed. The first method applies a second order Taylor series to calculate the POD coefficients at each time step from the POD coefficients at earlier time steps. The second method uses a Smolyak sparse grid collocation method to calculate the POD coefficients, where again the coefficients at earlier time steps are used as the inputs. The advantage of both approaches are that they are non-intrusive and so do not require modifications to a system code; they are therefore very easy to implement. They also provide accurate solutions for modelling flow problems, and this has been demonstrated by the simulation of flows past a cylinder and within a gyre. It is demonstrated that accuracy relative to the high fidelity model is maintained whilst CPU times are reduced by several orders of magnitude in comparison to high fidelity models.


*Keywords:* Non-intrusive model reduction; Smolyak sparse grid; Taylor series; POD; Navier–Stokes

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## 1. Introduction

Reduced order models (ROMs) have become prevalent in many fields of physics as they offer the potential to simulate dynamical systems with substantially increased computation efficiency in comparison to standard techniques. Among the model reduction techniques, the proper orthogonal decomposition (POD) method has proven to be an efficient means of deriving a reduced basis for high-dimensional nonlinear flow systems. The POD method has been successfully applied to numerous research fields and has a number of variants, such as the principal component analysis (PCA) method [1] in statistics; Karhunen–Loeve method [2] in signal analysis and pattern recognition; and empirical orthogonal functions (EOF) [3,4] in geophysical fluid dynamics and meteorology. The POD method has also been applied to ocean models in Cao et al. [5], Vermeulen and Heemink [6] and also to shallow water equation models. Its application includes the work of Daescu and Navon [7], Stefanescu et al. [8,9], Chen et al. [10,11], Altaf et al. [12], Du et al. [13], Fang et al. [14], as well as Xiao et al. [15,16].

However in most cases the source code describing the physical model has to be modified in order to generate the reduced order model. These modifications can be complex, especially in legacy codes, or may not be possible if the source code is not available (*e.g.* in some commercial software) [17]. To circumvent these shortcomings, more recently, non-intrusive methods have been introduced into ROMs, which do not require the knowledge of the governing equations and the original code [17]. Noack [18] and Noori [19] introduced the Neural Network into ROMs. Chen [17] proposed a Black Box Stencil interpolation non-intrusive method, which is based on parametric regression methods, and applied it to a one dimensional chemical reaction problem and two dimensional porous media flow problems. Audouze et al. [20] proposed a non-intrusive Radial Basis Function (RBF) reduced-order modelling method for approximating the solutions of nonlinear time-dependent parameterized partial differential equations (Burgers' equation and a parameterized convection–reaction–diffusion problem). Iuliano and Quagliarella [21] developed a non-intrusive POD ROM for aerodynamic shape optimization. Guénot et al. [22], Casenave et al. [23] and Klie [24] proposed a non-intrusive POD ROM based on RBF and the EIM/DEIM algorithm. However, most of current non-intrusive ROMs may still suffer from prohibitive computational costs due to the exponential increase of the number of multidimensional functions with the dimensional size of problems (in ROM, the dimensional size $d = P \times N_v$, where $P$ is the number of POD bases and $N_v$ is the number of variables to be solved).

To cope with the curse of dimensionality, as we know, the Smolyak sparse grid method [25] is an efficient method of integrating/interpolating multidimensional functions based on a univariate quadrature rule. This sparse grid method has been widely applied in various applications [26–28], including numerical integration [29], partial differential equations [30], economics [31,32], stochastic natural convection problems [33], sensitivity analysis [34], portfolio problems [35] and high dimensional interpolation [36].

To our best knowledge, little attempt has been made to use the sparse grid method in ROMs with exception of Peherstorfer [37], Cheng [38], Ullmann [39] and Lang, and Sumant [40]. Peherstorfer [37] presented a reduced-order model of parameterized systems by employing a sparse grid machine learning method and applied this new ROM to thermal conduction and chemical reaction simulations. Sumant [40] used a Smolyak algorithm to compute orthogonal polynomial expansions coefficients in the reduction of random input variables for an electromagnetic problems. Cheng [38] presented a method for numerical simulation of the stochastic Berger equation, and investigated the sparsity property in terms of Karhunen–Loeve expansions. Ullmann [39] and Lang assessed the applicability of POD/Galerkin to stochastic collocation on the sparse grid.

This paper presents the first work to apply non-intrusive ROMs to the Navier–Stokes equations. These ROMs are implemented here within a high fidelity unstructured mesh fluid model. The ability of non-intrusive ROMs to capture the highly nonlinear fluid dynamics is investigated here. The first non-intrusive ROM uses a sparse grid collocation approach (based on Smolyak grids) and another is derived using Taylor series expansion. The reduced order models are constructed using a finite element Bubnov–Galerkin discretization of the Fluidity fluid dynamics modelling software [41] taking snapshots of the solution variables at regular time intervals. In the Smolyak sparse grid ROM approach, solutions of the full model are recorded (as a sequence of snapshots), and from this data appropriate basis functions are formed that optimally represent the problem. The Smolyak sparse grid method is used to construct interpolation functions that approximate the non-linearity of the model. In the Taylor/POD approach, the model based on snapshots is expanded through a Taylor expansion to second order so as to capture the quadratic non-linearities in the high fidelity system.