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A Petrov–Galerkin finite element method for variable-coefficient fractional diffusion equations

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Abstract

Fractional diffusion equations have found increasingly more applications in recent years but introduce new mathematical and numerical difficulties. Galerkin formulation, which was proved to be coercive and well-posed for fractional diffusion equations with a constant diffusivity coefficient, may lose its coercivity for variable-coefficient problems. The corresponding finite element method fails to converge.

We utilize the discontinuous Petrov–Galerkin (DPG) framework to develop a Petrov–Galerkin finite element method for variable-coefficient fractional diffusion equations. We prove the well-posedness and optimal-order convergence of the Petrov–Galerkin finite element method. Numerical examples are presented to verify the theoretical results.

Keywords: Discontinuous Petrov–Galerkin framework; Fractional diffusion equations; Petrov–Galerkin finite element method; Weak coercivity

1. Introduction

In the last few decades fractional differential equations (FDEs) have found increasingly more applications in fluid mechanics [1], anomalous diffusion and acceleration of steep fronts in reaction–diffusion processes [2,3], turbulence in geophysical flows or plasma physics [4–6], continuum mechanics [7], as they provide very effective alternatives for modeling complex systems characterized by nonlocal phenomena and long range interactions. However, FDEs present mathematical difficulties that have not been encountered in the context of second-order differential equations. In their pioneer work [8], Ervin and Roop proved coercivity of a Galerkin formulation and the well-posedness of the homogeneous Dirichlet boundary-value problem of a constant–coefficient conservative FDE. We showed that for variable-coefficient FDEs the Galerkin formulation loses its coercivity [9] and that the Galerkin finite element methods might fail to converge [10].

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To overcome these difficulties we proposed a Petrov–Galerkin formulation for the homogeneous Dirichlet boundary-value problem of FDEs, and proved its weak coercivity and well-posedness [9]. However, there is a sharp difference between a Galerkin formulation and a Petrov–Galerkin formulation: Coercivity of a Galerkin formulation on an infinite-dimensional admissible space ensures that of the formulation on any finite-dimensional subspace. Consequently, the unique solvability and stability of Galerkin finite element methods are guaranteed automatically. In contrast, weak coercivity of a Petrov–Galerkin formulation on a pair of infinite-dimensional product spaces cannot ensure that of the formulation on any pair of finite-dimensional subspaces. Therefore, one still has to analyze how to choose appropriate finite-dimensional trial space and test space to ensure the weak coercivity and so the unique solvability and stability of the corresponding Petrov–Galerkin finite element method.

In this paper we utilize the DPG (discontinuous Petrov–Galerkin) framework of Demkowicz and Gopalakrishnan [11–14] to develop a Petrov–Galerkin finite element method for a class of variable-coefficient conservative FDEs in one space dimension. We prove its error estimate in the energy norm and the L^2 norm. Numerical experiments are presented to verify the convergence rates of the method. The rest of the paper is organized as follows: In Section 2 we present the model problem and cite known results to be used subsequently. In Section 3 we apply the DPG framework to the model problem. In Section 4 we develop a Petrov–Galerkin finite element method with optimal test functions for fractional diffusion equations with a constant diffusivity coefficient. We then prove the corresponding error estimates. In Section 5 we develop a Petrov–Galerkin finite element method with approximately optimal test functions for fractional diffusion equations with a variable diffusivity coefficient and prove the corresponding error estimates in the energy norm and the L^2 norm. In Section 6 we conduct numerical experiments to investigate the performance of the Petrov–Galerkin method and to verify its convergence rate numerically. In Section 7 we draw concluding remarks and outline future work.

2. Problem formulation

Let $C_0^\infty(0,1)$ be the space of infinitely many times differentiable functions on $(0,1)$ that are compactly supported within $(0,1)$. Let $L^p(0,1)$, with $1 \le p \le +\infty$, be the standard normed spaces of pth power Lebesgue integrable functions on $(0,1)$. Let $W^{m,p}(0,1)$ be the Sobolev space of functions on $(0,1)$ whose weak derivatives up to order m are in $L^p(0,1)$. Let $H^\mu(0,1)$, with $\mu > 1/2$, be the fractional Sobolev space of order μ and $H_0^\mu(0,1)$ be the completion of $C_0^\infty(0,1)$ with respect to the Sobolev norm $\|\cdot\|_{H^\mu(0,1)}$. Let $H^{-\mu}(0,1)$ be the dual space of $H_0^\mu(0,1)$.

We consider the variable-coefficient conservative FDE in one space dimension

$$-D\big[K(x)\big(\theta\, {}_0^C D_x^{1-\beta} u - (1-\theta)\, {}_x^C D_1^{1-\beta} u\big)\big] = f(x), \quad x \in (0,1), \qquad u(0) = u(1) = 0. \tag{1}$$

Here $Du(x) := u'(x)$ is the first-order differential operator, $2-\beta$ with $0<\beta<1$ represents the order of anomalous diffusion of the problem, K is the diffusivity coefficient with

$$0 < K_{\min} \le K(x) \le K_{\max} < \infty, \tag{2}$$

$0 \le \theta \le 1$ indicates the relative weight of forward versus backward transition probability, and f is the right-hand side. The left and right fractional integrals of order β are defined for any function $u \in L^p(0,1)$ by [15,16]

$${}_0D_x^{-\beta} u(x) := \frac{1}{\Gamma(\beta)} \int_0^x \frac{u(s)}{(x-s)^{1-\beta}} ds, \qquad {}_xD_1^{-\beta} u(x) := \frac{1}{\Gamma(\beta)} \int_x^1 \frac{u(s)}{(s-x)^{1-\beta}} ds,$$

where $\Gamma(\cdot)$ is the Gamma function. The left and right Caputo and Riemann–Liouville fractional derivatives of order β are defined by

$$\begin{aligned} {}_0^C D_x^\beta u(x) &:= {}_0D_x^{-(1-\beta)} Du(x), \qquad & {}_x^C D_1^\beta u(x) &:= -{}_xD_1^{-(1-\beta)} Du(x),\\ {}_0^R D_x^\beta u(x) &:= D\, {}_0D_x^{-(1-\beta)} u(x), \qquad & {}_x^R D_1^\beta u(x) &:= -D\, {}_xD_1^{-(1-\beta)} u(x). \end{aligned}$$

The mathematical model (1) arises in many physical and engineering applications. In groundwater hydrology, (1) represents the pressure equation for the flow, in which u is the water head, $K(x)$ is the intrinsic permeability of the porous medium, and f is the source and sink term [17]. Eq. (1) is obtained by incorporating a fractional Darcy's law, which accounts for the non-local interaction in the flow, into a mass balance law for the flow [18]. In the context of