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Reliable and efficient a posteriori error estimation for adaptive IGA boundary element methods for weakly-singular integral equations

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Abstract

We consider the Galerkin boundary element method (BEM) for weakly-singular integral equations of the first-kind in 2D. We analyze some residual-type a posteriori error estimator which provides a lower as well as an upper bound for the unknown Galerkin BEM error. The required assumptions are weak and allow for piecewise smooth parametrizations of the boundary, local mesh-refinement, and related standard piecewise polynomials as well as NURBS. In particular, our analysis gives a first contribution to adaptive BEM in the frame of isogeometric analysis (IGABEM), for which we formulate an adaptive algorithm which steers the local mesh-refinement and the multiplicity of the knots. Numerical experiments underline the theoretical findings and show that the proposed adaptive strategy leads to optimal convergence.

Keywords: Isogeometric analysis; Boundary element method; A posteriori error estimate; Adaptive mesh-refinement

1. Introduction

Isogeometric analysis. The central idea of isogeometric analysis is to use the same ansatz functions for the discretization of the partial differential equation at hand, as are used for the representation of the problem geometry. Usually, the problem geometry Ω is represented in computer aided design (CAD) by means of NURBS or T-splines. This concept, originally invented in [1] for finite element methods (IGAFEM) has proved very fruitful in applications [1,2]; see also the monograph [3]. Since CAD directly provides a parametrization of the boundary $\partial\Omega$, this makes the boundary element method (BEM) the most attractive numerical scheme, if applicable (i.e., provided that the fundamental solution of the differential operator is explicitly known). Isogeometric BEM (IGABEM) has first been considered for 2D BEM in [4] and for 3D BEM in [5]. Unlike standard BEM with piecewise polynomials which is well-studied in the literature, cf. the monographs [6,7] and the references therein, the numerical analysis of IGABEM is essentially open. We only refer to [2,8–10] for numerical experiments and to [11] for some quadrature analysis. In particular, a posteriori error estimation has been well-studied for standard BEM, e.g., [12–18] as well as the recent overview article [19], but

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has not been treated for IGABEM so far. The purpose of the present work is to shed some first light on a posteriori error analysis for IGABEM which provides some mathematical foundation of a corresponding adaptive algorithm.

Main result. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $\Gamma \subseteq \partial\Omega$ be a compact, piecewise smooth part of the boundary with finitely many connected components (see Sections 2.2 and 2.3). Given a right-hand side f, we consider boundary integral equations in the abstract form

$$V\phi(x) = f(x) \quad \text{for all } x \in \Gamma, \tag{1.1}$$

where $V : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an elliptic isomorphism. Here $H^{1/2}(\Gamma)$ is a fractional-order Sobolev space, and $\widetilde{H}^{-1/2}(\Gamma)$ is its dual (see Section 2). Given $f \in H^{1/2}(\Gamma)$, the Lax–Milgram lemma provides existence and uniqueness of the solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of the variational formulation of (1.1)

$$\int_\Gamma V\phi(x)\psi(x)\,dx = \int_\Gamma f(x)\psi(x)\,dx \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma). \tag{1.2}$$

In the Galerkin boundary element method (BEM), the test space $\widetilde{H}^{-1/2}(\Gamma)$ is replaced by some discrete subspace $\mathcal{X}_h \subseteq L^2(\Gamma) \subseteq \widetilde{H}^{-1/2}(\Gamma)$. Again, the Lax–Milgram lemma guarantees existence and uniqueness of the solution $\phi_h \in \mathcal{X}_h$ of the discrete variational formulation

$$\int_\Gamma V\phi_h(x)\psi_h(x)\,dx = \int_\Gamma f(x)\psi_h(x)\,dx \quad \text{for all } \psi_h \in \mathcal{X}_h, \tag{1.3}$$

and ϕ_h can in fact be computed by solving a linear system of equations.

We assume that $\mathcal{X}_h$ is linked with a partition $\mathcal{T}_h$ of Γ into a set of connected segments. For each vertex $z \in \mathcal{N}_h$ of $\mathcal{T}_h$, let $\omega_h(z) := \bigcup\{T \in \mathcal{T}_h : z \in T\}$ denote the node patch. If $\mathcal{X}_h$ is sufficiently rich (e.g., $\mathcal{X}_h$ contains certain splines or NURBS; see Section 4), we prove that

$$C_{\rm rel}^{-1}\,\|\phi - \phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \le \eta_h := \Big(\sum_{z\in\mathcal{N}_h} |r_h|^2_{H^{1/2}(\omega_h(z))}\Big)^{1/2} \le C_{\rm eff}\,\|\phi - \phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \tag{1.4}$$

with some $\mathcal{X}_h$-independent constants $C_{\rm eff}, C_{\rm rel} > 0$, i.e., the unknown BEM error is controlled by some computable a posteriori error estimator η_h. Here, $r_h := f - V\phi_h \in H^{1/2}(\Gamma)$ denotes the residual and

$$|r_h|_{H^{1/2}(\omega_h(z))} := \int_{\omega_h(z)}\int_{\omega_h(z)} \frac{|r_h(x) - r_h(y)|^2}{|x-y|^2}\,dy\,dx \tag{1.5}$$

is the Sobolev–Slobodeckij seminorm.

Estimate (1.4) has first been proved by Faermann [17] for closed $\Gamma = \partial\Omega$ and standard spline spaces $\mathcal{X}_h$ based on the arclength parametrization $\gamma : [0, L] \to \Gamma$. In isogeometric analysis, γ is *not* the arclength parametrization. In our contribution, we generalize and refine the original analysis of Faermann [17]: Our analysis allows, first, closed as well as open parts of the boundary, second, general piecewise smooth parametrizations γ and, third, covers standard piecewise polynomials as well as NURBS spaces $\mathcal{X}_h$.

Outline. Section 2 recalls the functional analytic framework, provides the assumptions on Γ and its parametrization γ, and fixes the necessary notation. The proof of (1.4) is given in Section 3 for sufficiently rich spaces $\mathcal{X}_h$ (Theorem 3.1). In Section 4, we recall the NURBS spaces for IGABEM and prove that these spaces $\mathcal{X}_h$ satisfy the assumptions (Assumptions (A1)–(A2) in Section 3.1) of the a posteriori error estimate (1.4). Based on knot insertion, we formulate an adaptive algorithm which is capable to control and adapt the multiplicity of the nodes as well as the local mesh-size (Algorithm 4.5). Section 5 gives some brief comments on the stable implementation of adaptive IGABEM for Symm's integral equation and provides the numerical evidence for the superiority of the proposed adaptive IGABEM over IGABEM with uniform mesh-refinement. In the final Section 6, some conclusions are drawn and some comments on future work and open questions are reported.

2. Preliminaries

The purpose of this section is to collect the main assumptions on the boundary and its discretization as well as to fix the notation. For more details on Sobolev spaces and the functional analytic setting of weakly-singular integral equations, we refer to the literature, e.g., the monographs [20,21,6] and the references therein.