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A posteriori error estimation for the fractional step theta discretization of the incompressible Navier–Stokes equations

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Highlights

- A goal-oriented a posteriori error estimator for time-discretization with the Fractional-Step Theta Method is introduced.
- We introduce a Petrov–Galerkin formulation, that is – up to a quadrature error – equivalent to the Fractional-Step Theta time-stepping scheme.
- This new error estimator is applied to nonstationary incompressible flow problems.

Abstract

In this work, we derive a goal-oriented a posteriori error estimator for the error due to time discretization. As time discretization scheme we consider the fractional step theta method, that consists of three subsequent steps of the one-step theta method. In every sub-step, the full incompressible system has to be solved (in contrast to time integrators of operator splitting type). The resulting fractional step theta method combines various desirable properties like second order accuracy, strong A-stability and very little numerical dissipation.

The derived error estimator is based on a mathematical trick: we define an intermediate time-discretization scheme based on a Petrov–Galerkin formulation. This method is up to a numerical quadrature error equivalent to the theta time stepping scheme. The error estimator is assembled as one weighted residual term given by the Dual Weighted Residual method measuring the error between real solution and solution to the Petrov–Galerkin formulation (that at no time has to be calculated) and one additional residual estimating the discrepancy between actual time stepping scheme used for simulation and the intermediate Petrov–Galerkin formulation.
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1. Introduction

In this work, we develop a technique for goal-oriented error estimation and time step adaption for the incompressible Navier–Stokes equations, discretized with the fractional step theta method. This time stepping method, introduced

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by Glowinski et al. [1,2] is one of the most attractive time discretization schemes for incompressible flows, as it combines several attributes like strong A-stability, very little numerical dissipation and second order accuracy. To avoid confusion, we note that we are not applying the fractional step theta method as an operator splitting scheme. One step $t_{m-1} \to t_m$ of the fractional step theta method is combined from three substeps of the one-step theta scheme, here formulated for the time-discretization of a simple initial value problem $u'(t) = f(t, u(t))$:

$$u_k^m - u_k^{m-1} = \theta k f(t^m, u_k^m) + (1-\theta) k f(t^{m-1}, u_k^{m-1}).$$

Applied to the Navier–Stokes equations, one step of the fractional step theta method requires three sub-steps with the full saddle point system.

For such a traditional finite difference approximation, a posteriori error control and time step adaption are usually based on estimation of the truncation error. In [3], the authors presented a new technique of error estimation for the theta and fractional theta method for parabolic problems. This error estimator is based on the following idea: The discrete solution u_k^m for $m = 1, 2, \ldots, M$ is calculated with the theta time stepping method. As we cannot directly estimate the error $u(t_m) - u_k^m$ between the numerical solution and the real solution, we introduce an intermediate problem based on a Galerkin formulation of a variational problem of the Navier–Stokes equations. This Galerkin solution is referred to as u_k^G. Now, the discretization error can be split by the triangle inequality

$$|u(t_m) - u_k^m| \le |u(t_m) - u_k^G(t_m)| + |u_k^G(t_m) - u_k^m|,$$

and two separate errors must be estimated. Estimating the first contribution is a standard task for Galerkin methods and can be accomplished by residual estimation. The second error contribution turns out to be given by a numerical quadrature error, that describes the difference between the time stepping method and the Galerkin method. Finally, we note out, that the Galerkin scheme is only used as mathematical construct. At no time it will be necessary to compute or approximate its solution u_k^G.

For error estimation, the goal-oriented approach going back to Becker and Rannacher [4,5] is employed. In particular considering problems in fluid-dynamics, one is very often interested in technical quantities like drag coefficients. By goal-oriented error estimation we can estimate the error and optimize the mesh specifically with respect to such functionals.

In this article, we generalize the error estimation technique described in [3] for parabolic problems to the incompressible Navier–Stokes equations by a suitable treatment of the divergence-free condition.

The following two sections will give the mathematical setting and background of the error estimator. The focus of Section 2 will be a Galerkin formulation for the incompressible Navier–Stokes equations similar to the theta method. Then, in Section 3 we derive the a posteriori error estimator based on this Galerkin formulation. In Section 4, we summarize the error estimator and describe the adaptive procedure as a computational approach usable in applications. In Section 5, we present numerical examples, demonstrating the accuracy of the error estimator and the efficiency of locally refined time discretizations.

2. Time-Galerkin discretizations of the Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be a two ($d = 2$) or three ($d = 3$) dimensional domain with smooth or convex polygonal boundary and $I = (0, T)$ be a time interval. By $L^2(\Omega)$ we denote the Lebesgue space of square integrable functions and by $H_0^1(\Omega; \Gamma)$ the Sobolev space of square integrable functions with first weak derivatives in $L^2(\Omega)$ and trace zero on (parts of) the boundary $\Gamma \subset \partial\Omega$, with $H_0^1(\Omega) := H_0^1(\Omega; \partial\Omega)$. Further, we introduce the Hilbert spaces $\boldsymbol{X}^v$ and X^p defined as

$$\boldsymbol{X}^v = \{\boldsymbol{v} \mid \boldsymbol{v} \in L^2(I, \boldsymbol{V}) \quad \text{and} \quad \partial_t \boldsymbol{v} \in L^2(I, \boldsymbol{V}^*)\}, \qquad X^p = L^2(I, H),$$

with $\boldsymbol{V} = H_0^1(\Omega; \Gamma)^d$, $H = L^2(\Omega)$, and the dual space $\boldsymbol{V}^*$ of $\boldsymbol{V}$. We denote the pair of the spaces $\boldsymbol{X}^v$ and X^p by $\mathcal{X} = \boldsymbol{X}^v \times X^p$. The inner product of $L^2(I, H)$ and the duality pairing between $L^2(I, \boldsymbol{V})$ and $L^2(I, \boldsymbol{V}^*)$ are defined as

$$(p, q)_I := \int_I (p(t), q(t))_H \,\mathrm{d}t \quad \text{and} \quad \langle \boldsymbol{u}, \boldsymbol{v} \rangle_I := \int_I \langle \boldsymbol{u}(t), \boldsymbol{v}(t) \rangle_{\boldsymbol{V} \times \boldsymbol{V}^*} \,\mathrm{d}t.$$

The extension of $(\cdot, \cdot)_I$ to function from $L^2(I, H^d)$ is straightforward.

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