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Goal-oriented error estimation based on equilibrated-flux reconstruction for finite element approximations of elliptic problems

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Abstract

We propose an approach for goal-oriented error estimation in finite element approximations of second-order elliptic problems that combines the dual-weighted residual method and equilibrated-flux reconstruction methods for the primal and dual problems. The objective is to be able to consider discretization schemes for the dual solution that may be different from those used for the primal solution. It is only assumed here that the discretization methods come with a priori error estimates and an equilibrated-flux reconstruction algorithm. A high-order discontinuous Galerkin (dG) method is actually the preferred choice for the approximation of the dual solution thanks to its flexibility and straightforward construction of equilibrated fluxes. One contribution of the paper is to show how the order of the dG method for asymptotic exactness of the proposed estimator can be chosen in the cases where a conforming finite element method, a dG method, or a mixed Raviart–Thomas method is used for the solution of the primal problem. Numerical experiments are also presented to illustrate the performance and convergence of the error estimates in quantities of interest with respect to the mesh size.

Keywords: Finite element method; Goal-oriented estimates; Quantity of interest; Dual problem; Asymptotically-exact error estimates

1. Introduction

A variety of finite element discretization schemes, such as the mixed finite element methods, the non-conforming finite element methods, or discontinuous Galerkin (dG) finite element methods, have been developed during the past decades in order to provide better approximation properties than those offered by the classical finite element method depending on the type of partial differential equations at hand. These methods have become increasingly popular and are now widely used for solving problems of various interest in engineering and sciences. At the same time, the necessity to obtain accurate finite element approximations to given boundary-value problems has stimulated the development of *a posteriori* error estimators that provide fully computable, reliable, and efficient error bounds in terms

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of the problem data and the finite element approximation. In the case of standard conforming finite element schemes, *a posteriori* error estimates with respect to global energy norms are indeed well-established, see e.g. [1–8]. *A posteriori* error estimation for non-conforming finite element methods with application to second-order elliptic problems has recently seen significant progress and is still the subject of sustained research efforts, see e.g. [9–19] for dG methods and [20–24] for mixed finite element methods. We also refer the reader to [15,25–28] and references therein for the presentation of unifying frameworks on the topic.

In practical applications, end users are however interested in error estimates in some specific features of the true solution. These so-called quantities of interest, either local or global, are represented as functionals defined on the vector space of trial solutions of the boundary-value problem. Error estimation with respect to such functionals is usually referred to as goal-oriented error estimation. The key ingredient for goal-oriented error estimation is the formulation of an auxiliary problem, the dual problem to the primal problem, whose solution provides necessary information for reliable estimates of the error in the goal functional. Several strategies for goal-oriented error estimation have been proposed in the case of elliptic problems: goal-oriented error estimates based on energy norm of the errors in the primal and dual solutions were introduced in [29–32] and further developed by various authors, see for example [2,33], and references therein, error estimates using the dual-weighted residual method were proposed in [4,34,35]; functional a posteriori error estimates were developed in [5,7]; estimates based on the gradient-recovery method were considered in [36–40]; goal-oriented estimates for discontinuous Galerkin methods in the case of second-order elliptic problems were derived in [41]. Finally, goal-oriented error estimators, based on p-refinement using hierarchical smooth (isogeometric and B-spline) basis functions for the solution of the dual problem, were considered in [42,43].

The general approach to obtain goal-oriented error estimates consists, on one hand, in deriving an error representation involving the residual functional and the exact dual (adjoint) solution, and, on the other hand, in constructing a sufficiently accurate approximation of the adjoint solution in order to obtain a fully computable and reliable error estimate along with local refinement indicators. In the case of the classical conforming finite element method, such approximation to the adjoint solution is usually calculated on a refinement of the mesh used for the primal solution, with the same polynomial degree, or, preferably, on the same mesh, but with a higher polynomial degree.

In this paper, we propose an alternative approach to goal-oriented estimation by considering an error representation that does not use the orthogonality property and is amenable to different types of discretization of the primal and dual problems. We only suppose here that the method used to discretize the primal problem produces piecewise polynomial solutions from which equilibrated fluxes can be reconstructed (this is not a restrictive property) and satisfies standard a priori error estimates in the L^2 and energy norms. Using an error representation similar to the one used in the dual-weighted residual method, we use the flux-equilibration technique to decompose the error into a computable error estimator and a higher-order remainder. It is well known, see e.g. [33], that in order to obtain an efficient error estimator (effectivity indices remain close to unity), the dual problem must be approximated using a higher-order approximation than the one used in the finite element approximation of the primary problem. We propose in this paper, in order to approximate the dual solution, that a high-order discontinuous Galerkin method be used and applied on the same mesh as that used for the discretization of the primal problem. The choice of the dG method seems natural owing to its flexibility in using non-uniform high-order polynomials. Furthermore, the dG method benefits from the fact that it is locally conservative; it implies that the construction of equilibrated fluxes, needed for the evaluation of the proposed error estimator, is rather straightforward, see e.g. [25,44–46]. Finally, we show that, depending on the approximation properties of the primal method and problem data, the order of the dG method, used to solve the dual problem, can be chosen in such a way that the respective error estimator is asymptotically exact.

We note here that the suggested approach to error analysis with respect to quantities of interest can be extended to solutions of non-linear elliptic problems and to non-linear convection–diffusion problems. The principal ingredients of the proposed techniques, i.e. equilibrated-flux reconstruction and high-order discontinuous Galerkin approximations of the dual problem, can be adapted to non-linear convective and diffusive fluxes in stabilized versions of the discontinuous Galerkin methods for convective-diffusion problems. Similar techniques can also be used for time-dependent convective-diffusion problems for which the DG method is used for discretization in time and space.

The paper is organized as follows. Section 2 introduces the model (primal) problem, the corresponding dual problem, and preliminary notation for goal-oriented error estimation. Section 3 presents a suitable error representation for the goal functional based on the use of equilibrated fluxes in terms of the finite element solutions to the primal and dual problems. We also show how the error representation can be decomposed into a fully computable error estimator and a higher-order term, which can be evaluated using (1) a priori estimates with respect to the primal and dual

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