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Bases of T-meshes and the refinement of hierarchical B-splines

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Abstract

In this paper we consider spaces of bivariate splines of bi-degree (m, n) with maximal order of smoothness over domains associated to a two-dimensional grid. We define admissible classes of domains for which suitable combinatorial technique allows us to obtain the dimension of such spline spaces and the number of tensor-product B-splines acting effectively on these domains. Following the strategy introduced recently by Giannelli and Jüttler, these results enable us to prove that under certain assumptions about the configuration of a hierarchical T-mesh the hierarchical B-splines form a basis of bivariate splines of bi-degree (m, n) with maximal order of smoothness over this hierarchical T-mesh. In addition, we derive a sufficient condition about the configuration of a hierarchical T-mesh that ensures a weighted partition of unity property for hierarchical B-splines with only positive weights.

Keywords: T-mesh; Spline space; Local refinement; Hierarchical B-splines

1. Introduction

The spline representations using T-mesh as an underlying structure have absorbed substantial interest among designers for the last decade. The basic motivation to apply such representations in design and analysis is to break tensor-product structure of geometric representation used in NURBS. A new interest in this issue has emerged recently in connection with isogeometric analysis, see Cottrell et al. [1]. In this paper we deal with the concept of splines over T-meshes stated originally by Deng et al. [2]. The issue of describing splines over a general T-mesh seems hardly solvable. In order to be able to generate spline basis functions and refine a spline space, we need to restrict ourselves on reasonable classes of T-meshes.

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For arbitrary TR-meshes, which include T-meshes, the dimension formula and basis functions have been derived for the C^0 case by Schumaker and Wang [3]. For the case of bivariate splines of bi-degree (m, n) with the reduced order of smoothness (r, r'), i.e. $m \geqslant 2r + 1 > 0$ and $n \geqslant 2r' + 1 > 0$, spline basis functions have been obtained in terms of Bernstein–Bézier coefficients for T-meshes without cycles [4]. We note that this class of T-meshes includes the natural ones obtained as a result of refining a given rectangle by successive splitting rectangles into two subrectangles. For hierarchical T-meshes, the construction of PHT-splines [5], which are splines of bi-degree (3, 3) and the order of smoothness (1, 1), showed an efficiency in surface modeling and isogeometric analysis.

The construction of splines of bi-degree (m, n) with the order of smoothness (r, r') becomes more sophisticated for understanding when $m < 2r+1$ and $n < 2r'+1$. It is worthwhile to analyze the class of hierarchical T-meshes for which the hierarchical B-splines, showing already their efficiency in applications, provide a basis of a spline space. Hierarchical B-splines for surface modeling were originally introduced by Forsey and Bartels [6]. Kraft [7] suggested a selection mechanism for hierarchical B-splines that ensures their linear independence as well as local refinement control. Vuong et al. [8] considered applications of hierarchical B-splines in isogeometric analysis.

Giannelli and Jüttler [9] have recently proved that for a hierarchical T-mesh, determined by a nested sequence of domains $\Omega^0 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \emptyset$ associated with a nested sequence of grids $G^0 \subset \cdots \subset G^{N-1}$, the hierarchical B-splines span the space of splines of bi-degree (m, m) with maximal order of smoothness if each domain $R^\ell = \Omega^0 \setminus \Omega^\ell, \ell = 1, \ldots, N$, considered with respect to the grid $G^{\ell-1}$, lays in a certain class. Later, that result has been generalized for splines of tri-degree (m, m, m) [10].

In this paper we extend the results from [9,10] to the case of splines of bi-degree (m, n) with maximal order of smoothness. This extension requires new definition of admissible classes of domains associated with a two-dimensional grid. We define these classes inductively and in a purely combinatorial way. For a given bi-degree (m, n), a two-dimensional grid and a domain from an admissible class, we obtain the dimension of the spline space over this domain and the number of tensor-product B-splines acting effectively on it; furthermore, it appears that these two numbers are equal. Then, following the approach used in [9], we prove that for certain assumptions about the configuration of a hierarchical T-mesh the hierarchical B-splines form a basis of bivariate splines of bi-degree (m, n) with maximal order of smoothness over this hierarchical T-mesh. Also, we find an additional condition about the configuration of a hierarchical T-mesh that ensures a weighted partition of unity property for hierarchical B-splines with only positive weights.

The rest of this paper is organized as follows. In Section 2 we consider the basic one-dimensional case and prove propositions necessary for Section 3 where the two-dimensional case is considered. For given integers $k_1, k_2 \geqslant 0$ in Section 3.1 we introduce the class $\mathcal{A}^2_{k_1,k_2}$ of two-dimensional domains formed by the cells of an infinite two-dimensional grid. In Section 3.2 we derive the dimension for the space of tensor-product splines of bi-degree (m, n) with maximal order of smoothness defined on a domain from the class $\mathcal{A}^2_{m-1,n-1}$. In Section 3.3 we show that a basis of this space can be obtained as the set of tensor-product B-splines acting effectively on the domain. In Section 4, with the tools obtained in Section 3, we provide a condition on the configuration of a hierarchical T-mesh to guarantee that hierarchical B-splines span the space of splines of bi-degree (m, n) over this T-mesh (see Theorem 1). In addition, in Corollary 9 we present a condition on a hierarchical T-mesh ensuring the existence of a weighted partition of unity for hierarchical B-splines, with only positive weights. We conclude this paper with several remarks in Section 5.

2. Univariate case

Let T' be an infinite one-dimensional grid. For the sake of simplicity, we suppose that the distances between adjacent grid nodes of T' are equal to 1. A cell of T' is a closed segment of a length 1 between adjacent grid nodes. Let T'_1 be the grid that is obtained by shifting T' by $\frac{1}{2}$.

Let Ω be a closed bounded domain formed by a finite number of cells of T'. Then, Ω consists of a number of segments of finite length. A vertex of a domain Ω is a grid node of T' that belongs to Ω. We say that a vertex of Ω is an inner vertex if it lies in the interior of Ω, which is hereinafter denoted by int Ω. For a given domain Ω we define the dilatation domains Ω^e_k in a recursive manner:

Definition 1. If $k = 0$, $\Omega^e_0 := \Omega$. If $0 < k$ is odd, Ω^e_k is the union of the cells of T'_1 with vertices of Ω^e_{k-1} as their centroids. If $0 < k$ is even, Ω^e_k is the union of the cells of T' with vertices of Ω^e_{k-1} as their centroids.