Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 283 (2015) 1493–1524
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Variational multiscale a posteriori error estimation for systems: The Euler and Navier–Stokes equations

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Received 31 December 2013; received in revised form 16 October 2014; accepted 19 October 2014
Available online 4 November 2014

Abstract

This paper extends *explicit* a posteriori error estimators based on the variational multiscale theory to systems of equations. In particular, the emphasis is placed on flow problems: the Euler and Navier–Stokes equations. Three error estimators are proposed: the standard, the naive and the upper bound. Numerical results show that with a very economical algorithm the attained global and local efficiencies for the naive approach are reasonably close to unity whereas the standard and upper bound approaches give, respectively, approximate lower and higher error estimates.

Keywords: A posteriori error estimation; Euler equations; Navier–Stokes equations; Stabilized methods; Variational multiscale method

1. Introduction

In computational fluid dynamics, most of the successful a posteriori error estimators are *implicit* methods and, therefore, require solving additional partial differential equations at the global or local level [1]. One of the most used auxiliary equations is the *dual* or *adjoint* problem, because it indicates the influence of the error on the numerical solution (see [2–5] and references therein). Examples of such methods for the linear transport equation and stabilized methods can be found in [6,7], for hyperbolic systems in [8–11] and for the Euler and Navier–Stokes equations solved with a finite volume method in [12–14].

Other implicit strategies are based on solving primal local problems, like the *element* and *subdomain residual* methods. The element residual method is applied to estimate the error of the incompressible Navier–Stokes equations by [15–17] and to symmetrizable systems and the compressible Navier–Stokes equations by [18,19]. The subdomain residual method proposed in [20] estimates the error of the incompressible Navier–Stokes equations, where local primal problems are solved in patches of elements. Recently, based on the variational multiscale theory, various papers propose to compute explicitly the subgrid scales (or error) on patches of elements [21,22] or inside each element with an enriched basis [23]. These last three papers are applied to elliptic problems.

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http://dx.doi.org/10.1016/j.cma.2014.10.032

Although implicit methods are the most accurate methods, they are expensive. *Explicit* a posteriori error estimators try to overcome the cost associated with implicit methods.

Some milestones in the development of explicit residual-based techniques for Navier–Stokes equations and transport problems were set by Johnson and coworkers [24–27]. Also Verfürth derived an explicit error estimator for the Stokes problem [28,29], which is based on the computation of residuals. For the incompressible Navier–Stokes equations, [30] presents a residual-based method for solutions computed with a stabilized method. A very popular technique for error estimation which does not require solving additional equations is that based on super-convergence properties of recovered derivatives, such as in [31,32], which present recovery-based methods for incompressible flows. Other explicit methods can be considered *error indicators*, like [33], where the curvature or second derivative of the solution is used to adapt the mesh size of compressible flow calculations. In this line, for the incompressible Navier–Stokes equation it can be mentioned the error indicator [34]. An example for anisotropy mesh adaptation for the shallow water equations can be found in [35].

The strength of explicit methods resides in their computational economy. However, explicit methods are inaccurate or depend on unknown constants whose evaluation again requires solving local problems.

Although the present methodology cannot replace the adjoint approach, recently the variational multiscale theory [36] has shown promise in reconciliating economy and accuracy of a posteriori error estimation. In particular, it has been shown that the stabilizing parameter possesses accurate information about the global and local discretization error [37–42], a property that was hinted in [43]. In fact, the basic smooth error estimator has been applied to adapt the mesh for incompressible flows in [44] and similar ideas are being employed to model the small scales of turbulence fluctuations [45].

Indeed, the stabilizing parameter stems from solving *a priori* the dual problem at the element level, which is the base of the accurate implicit methods. Furthermore, if the method has a local error distribution, the *local* dual problem has most of the error information and proper error time-scales can be derived from the *element* Green's function. As shown in [46], this is approximately the case of stabilized methods for the transport equation, and it will be assumed here that this property holds for more complex equations, minimizing the error propagation. Finally, the error estimator can be written in explicit form, yielding a computationally economical formulation to estimate the error accurately.

Following these ideas, in this paper the work established previously for the multi-dimensional transport equation is extended to linear systems of equations. The formulation is tested with the Euler and Navier–Stokes equations.

2. The variational multiscale approach to error estimation for linear systems

2.1. Strong form

Consider an open spatial domain Ω with boundary Γ, such that $\Gamma = \Gamma_{\mathcal{G}} \cup \Gamma_{\mathcal{H}}$. The strong form of the boundary-value problem consists of finding the solution vector $\boldsymbol{Y} : \Omega \to \mathbb{R}^{n_{\text{eq}}}$, where n_{eq} is the number of equations of the system (which coincides with the number of unknowns), such that for the given essential boundary conditions $\boldsymbol{\mathcal{G}} : \Gamma_{\mathcal{G}} \to \mathbb{R}^{n_{\text{eq}}}$ and the natural boundary conditions $\boldsymbol{\mathcal{H}} : \Gamma_{\mathcal{H}} \to \mathbb{R}^{n_{\text{eq}}}$, the following equations are satisfied

$$\begin{cases} \boldsymbol{\mathcal{L}Y} = \boldsymbol{0} & \text{in } \Omega \\ \boldsymbol{Y} = \boldsymbol{\mathcal{G}} & \text{on } \Gamma_{\mathcal{G}} \\ \boldsymbol{\mathcal{B}Y} = \boldsymbol{\mathcal{H}} & \text{on } \Gamma_{\mathcal{H}}. \end{cases} \tag{1}$$

More complex boundary conditions could be contemplated with simple extensions of the theory. Here $\boldsymbol{\mathcal{L}}$ represents a steady linear second order vector operator, such as the advective–diffusive system

$$\boldsymbol{\mathcal{L}Y} = \boldsymbol{A}_i \boldsymbol{Y}_{,i} + (\boldsymbol{K}_{ij} \boldsymbol{Y}_{,j})_{,j} - \boldsymbol{S} \tag{2}$$

where $\boldsymbol{A}_i$ are constant Euler Jacobians, $\boldsymbol{K}_{ij}$ constant diffusion matrices and $\boldsymbol{S}$ a linear source term. Many systems admit a conservative form, which can be expressed as [47,48]

$$\boldsymbol{\mathcal{L}Y} = \boldsymbol{F}^{\text{adv}}_{i,i} - \boldsymbol{F}^{\text{diff}}_{i,i} - \boldsymbol{S} \tag{3}$$

where $\boldsymbol{F}^{\text{adv}}_i$ is the ith advective flux with $\boldsymbol{A}_i = \boldsymbol{F}^{\text{adv}}_{i,\boldsymbol{Y}}$ and $\boldsymbol{F}^{\text{diff}}_i$, the ith diffusive flux, with $\boldsymbol{F}^{\text{diff}}_i = \boldsymbol{K}_{ij} \boldsymbol{Y}_{,j}$.