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A finite element pressure correction scheme for the Navier–Stokes equations with traction boundary condition

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Abstract

We consider the popular pressure correction scheme for the solution of the time dependent Navier–Stokes equations with *traction* boundary condition. A finite element based method to improve the performance of the classical approach is proposed. The improvement is achieved by modifying the traction boundary condition for the provisional velocity $\tilde{u}^{n+1}$ in each time step. The corresponding term consists of a simple boundary functional involving the normal derivative of the pressure correction that can be evaluated in a natural and easy way in the context of finite elements.

Computational results show a significant improvement of the solution, in particular for the pressure in the case of smooth domains.

MSC: 35K; 65M; 76D

Keywords: Navier–Stokes equations; Traction boundary condition; Open boundary condition; Pressure correction; Finite elements

0. Introduction

Because of their conceptual simplicity and low computational cost projection schemes are very popular methods to solve the unsteady incompressible Navier–Stokes equation. The history of these schemes dates back to the pioneering work of Chorin and Temam [1–3]. However, the price one usually has to pay for the simplicity of these schemes is a strong splitting error becoming manifest in an order reduction of the error. This is caused by the decoupling of velocity and pressure (that is at the core of the methods) resulting for instance in a non-physical behavior of the pressure close to the boundary [4]. Consequently much effort has been spent in removing or at least reducing this effect.

It is out of scope of this presentation to cite even the most relevant papers from the abundant literature on splitting or projection methods. Instead, we refer to the excellent overview paper [5] and the papers cited there for further reference.

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In this paper we are concerned with the time dependent Navier–Stokes equations, where a *traction boundary condition*

$$\sigma(u,p)\mathbf{n} = g \quad \text{on } \Gamma$$

is imposed on a part Γ of the boundary of the domain Ω. Here, $\sigma(u,p) = \frac{1}{\mathrm{Re}}D(u) - p\mathbf{I}$ denotes the stress tensor with the rate of strain tensor $D(u) = (\nabla u + \nabla u^T)$. Re is the Reynolds number and $\mathbf{n}$ the outward unit normal on Γ. The above type of boundary condition is important in itself for instance in certain technical applications and furthermore (maybe even more important) it constitutes the core problem in solving free boundary problems.

Note that replacing $D(u)$ by ∇u in the boundary condition results in a popular *open boundary* (or *do-nothing*) condition, see for instance [6–8]. This slightly simpler problem is also covered by our approach below.

Unlike for a fully coupled discretization, i.e. solving a saddle point problem in every time step, splitting schemes applied to flow problems with traction boundary conditions introduce much bigger errors than for Dirichlet boundary conditions. The reason behind may be seen in the fact that this type of boundary condition additionally couples velocity and pressure. Furthermore, despite its relevance, the question of splitting schemes for problems with traction boundary conditions is much less addressed than for Dirichlet conditions.

In this paper we introduce an improvement to the classical pressure correction schemes with traction boundary condition. Our approach is based on the following simple idea. Naturally, one usually imposes the following boundary condition for the provisional velocity $\tilde{u}$ at time instant t_{n+1}:

$$\sigma(\tilde{u}^{n+1}, p^n)\mathbf{n} = g^{n+1} \quad \text{on } \Gamma.$$

This choice, however, leads to an inconsistent boundary condition for the solution (u^{n+1}, p^{n+1}). Therefore we modify the above boundary condition by adding a still to be determined function l^{n+1} that hopefully leads to a more consistent boundary condition:

$$\sigma(\tilde{u}^{n+1}, p^n)\mathbf{n} = g^{n+1} + l^{n+1} \quad \text{on } \Gamma.$$

By some differential geometry calculus it is possible to determine l^{n+1} such that one even ends up with the *correct* boundary condition for (u^{n+1}, p^{n+1}). However, this would require again a fully coupled approach, since the correct l^{n+1} requires the new value of the pressure correction Φ^{n+1} that is not known at this stage of the scheme, see Section 3.2 below. Instead, an appropriate extrapolation for l^{n+1} can be used.

Let us mention some related work in the context of finite elements. The only rigorous error analysis for a pressure correction scheme with open boundary condition we are aware of is [9], see also Section 3.2. In [10] the Navier–Stokes equations in 2d with open boundary condition on a *straight* part of the boundary are considered. The approach is based on a Neumann-to-Dirichlet operator for the pressure in the context of the so called *unconstrained Navier–Stokes* equation approach introduced in [11,12]. Poux et al. [13] use an extrapolation for the boundary condition for the pressure correction Φ, again for the case of a *straight* part of the boundary of a 2d domain. This approach is somehow similar to ours, but differs even in the case of planar boundaries. Moreover, it is not obvious how to generalize the idea in [13] to curved boundaries.

The rest of this article is organized as follows. In Section 1 we state the problem and introduce some notation. Section 2 gives some results on differential operators on manifolds that are needed to develop our method. In Section 3 the new scheme is introduced. Computational results showing the improvement by the traction correction are discussed in Section 4. The paper is concluded by some final remarks in Section 5.

1. Notation and preliminaries

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be an open, connected and bounded domain with a sufficiently smooth traction boundary Γ. The approach and results of this paper hold, if Γ is a sub-manifold of $\partial\Omega$ without boundary and $\partial\Omega \setminus \Gamma$ is a Dirichlet boundary. For ease of presentation, however, in what follows, it is assumed that $\Gamma = \partial\Omega$. Consider the incompressible Navier–Stokes equation on a time interval $]0,T[$: find the velocity u and the pressure p fulfilling

$$\partial_t u + u \cdot \nabla u - \nabla \cdot \sigma(u,p) = f \quad \text{in } \Omega \times]0,T[,$$
$$\operatorname{div} u = 0 \quad \text{in } \Omega \times]0,T[,$$