Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 279 (2014) 227–246
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Stability of elastic rods with self-contact

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Received 8 February 2014; received in revised form 12 June 2014; accepted 17 June 2014
Available online 27 June 2014

Abstract

In this paper we study the numerical stability of equilibrium configurations of an elastic rod with unilateral frictionless self-contact. We use a finite-element method with penalty and augmented Lagrangian approaches to account for the self-penetration constraint. The numerical solution is carried out using the arclength continuation method. In the case where the energy expression satisfies some hypotheses, we have proved that any isolated minimum of the continuous self-contact problem is the strong limit of a sequence of local minima of the discrete problem. Our stability analysis is based on detecting the zeros of the determinant of a symmetrized version of the stiffness matrix and whether there is a sign change of the determinant or not.

MSC: 74K10; 74M15; 74S05

Keywords: Cosserat rods; Self-contact; Stability; Arclength continuation; Adapted augmented Lagrangian; Finite elements

1. Introduction

Some elastic structures must be designed to withstand large displacements and to be susceptible to self-contact. Such is the case for elastic rods undergoing large displacements and self-contact used as models of DNA in some biomedical research works [1–5]. The study of their equilibrium configurations is complicated by the rigorous modeling of the contact. In general, their mathematical formulation leads to a highly nonlinear problem whose solution suffers from non-uniqueness and instability.

The modeling of the self-contact in elastic rods has not been widely addressed especially in the case of rods undergoing bending, shear, torsion and elongation [6–8]. The main difficulty is that an elastic rod is represented by a curve which is a manifold of codimension two whereas most of the contact problems treated in the literature are for plates and shells that are represented by surfaces which are manifolds of codimension one [9–11].

The basic model that we are advocating here is the director theory of rods introduced by the Cosserat brothers [12], developed by Reissner [13–15] and Antman [16,17] and used by Simo and Vu-Quoc [18,19] and Ibrahimbegović

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http://dx.doi.org/10.1016/j.cma.2014.06.027

[10]. The contact model that we employ is the one we introduced in [20] which is based on the introduction of a contact distance. By taking into account the sign of this distance in the equilibrium equations we were able to prevent self-penetration of matter and identify contact points that are not known a priori.

For some geometrically-exact theories of rods, including the one we use here, the space of admissible configurations is not a linear space but rather a differential manifold. The tangent stiffness operator, typically computed by linearization of the weak form of the equilibrium equations, is generally nonsymmetric. Away from equilibrium configurations, the lack of symmetry reflects the effects of curvature of the differentiable manifold structure of the space of admissible configurations. At equilibrium configurations, however, the lack of symmetry is rather due to the nature of external loads and boundary conditions. In this case, the tangent stiffness operator is symmetric at equilibrium whenever the external forces are conservative [9]. In [11], Simo proposed an expression of a symmetric stiffness operator based on replacing the Gâteaux derivatives by covariant derivatives with respect to a Riemannian metric. The intrinsically obtained Hessian is then always symmetric and corresponds to the symmetry of the usual expression obtained by Gâteaux derivatives in case of linear spaces. For finite-element formulation, this common practice of symmetrization [11], must be justified since symmetry at the continuum level is not guaranteed at the discrete level.

The non-uniqueness of solution can be induced by the presence of stiff slopes on the equilibrium configuration curve. The difficulties arising from these slopes can generally be overcome by using an arclength continuation method. In such a method, a load factor is introduced to allow one to slowly follow a branch of solutions [21–23]. Despite its performance, the arclength continuation method may lose its efficiency when a critical point with the presence of a self-contact is reached. We recall that an equilibrium configuration is called stable if the change of energy is positive for all small perturbations and the equilibrium configuration is called unstable if the change of energy is negative for some small perturbations; such perturbations move the system to another equilibrium configuration with a lower energy. A critical point refers to a state of equilibrium where the rod passes from a stable configuration to an unstable configuration [21,22].

In this work, a stability analysis is presented and, because the consistent linearization used leads to a nonsymmetric tangent stiffness operator, a technique of symmetrization is discussed. In addition, adapted augmented Lagrangian and penalty are frequently used for the numerical treatment of the self-contact constraint. Its numerical implementation is based on the formulation of an extended system. Two kinds of stability analysis are studied in this paper, buckling and post-buckling i.e., the detection of critical points and the study of solutions beyond critical points. Our buckling study is based on the determination of zeros and detection of the sign changes of the determinant of the tangent stiffness matrix. For this, a bisection method was used. The tangent stiffness matrix associated with the discrete problem is not symmetric. In [24], the authors showed that ignoring the antisymmetric part of this matrix when solving the linear system of the Newton algorithm results in a quadratic convergence rate if the antisymmetric part is proportional to the residual, which is precisely the case of the stiffness matrices that we study.

The remainder of this paper is organized as follows. In Section 2 we give the mathematical formulation for the elastic rod model that will be used. The contact model is presented in Section 3. Section 4 is devoted to the finite-element formulation of the continuous problem and to study convergence of solutions of the discretized problem to a solution of the continuous problem. The details of the numerical methodology are described in Section 5. The numerical stability analysis is presented in Section 6. Finally, in Section 7 we present some numerical examples of stability analysis for rods with self-contact.

2. The mathematical formulation of the elastic rod problem

The basic model that we advocate here to describe the deformation of an elastic rod is the one introduced by the Cosserats [12], developed in the pioneering works of Reissner [13–15] and Antman [16,17] and exploited by Simo & Vu-Quoc [18] and Ibrahimbegović [10].

2.1. Kinematic

We consider an elastic rod $\mathscr{R}$ of length L and circular cross section of a uniform radius ε. The configuration of the rod is described by specifying, for each $s \in [0, L]$, a position vector $\boldsymbol{r}(s)$ and a right-handed triad of orthonormal directors $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$. The curve $\mathscr{C} \equiv \{\boldsymbol{r}(s), s \in [0, L]\}$ represents the centerline, in the deformed configuration of $\mathscr{R}$. The triad $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$ locates the orientation of the material cross section at s.