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Stabilized stress–velocity–pressure finite element formulations of the Navier–Stokes problem for fluids with non-linear viscosity

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Highlights

- Detailed description of the mixed three-field formulation of the incompressible Navier–Stokes equations, including non-Newtonian fluids.
- Derivation of mixed stabilized finite element methods for the three-field problem presented.
- Description of implementation issues of the formulations proposed.
- Numerical testing of the resulting formulation through numerous examples.

Abstract

The three-field (stress–velocity–pressure) mixed formulation of the incompressible Navier–Stokes problem can lead to two different types of numerical instabilities. The first is associated with the incompressibility and loss of stability in the calculation of the stress field, and the second with the dominant convection. The first type of instabilities can be overcome by choosing an interpolation for the unknowns that satisfies the appropriate inf–sup conditions, whereas the dominant convection requires a stabilized formulation in any case. This paper proposes two stabilized schemes of Sub-Grid Scale (SGS) type, differing in the definition of the space of the sub-grid scales, and both allowing to use the same interpolation for the variables $\boldsymbol{\sigma}$–$\boldsymbol{u}$–p (deviatoric stress, velocity and pressure), even in problems where the convection component is dominant and the velocity–stress gradients are high. Another aspect considered in this work is the non-linearity of the viscosity, modeled with constitutive models of quasi-Newtonian type. This paper includes a description of the proposed methods, some of their implementation issues and a discussion about benefits and drawbacks of a three-field formulation. Several numerical examples serve to justify our claims.

Keywords: Incompressible flows; Three-field formulation; Mixed finite element methods; Stabilized formulations; Non-Newtonian fluids

1. Introduction

Fluids, depending on their behavior under the action of shear-stress, can be classified as Newtonian and non-Newtonian. The last group is predominant in the petroleum industry, in chemical–pharmaceutical processes and in

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food products [1–4]. The non-Newtonian behavior is caused by the complex microstructure present in these fluids, in some cases originated by a mixture of different components, which in turn can be found in different states. For example, blood is a complex fluid composed of red cells, white cells, platelets and plasma, which at different shear stresses can pass from a fluid exhibiting Newtonian behavior to one exhibiting pseudoplastic non-Newtonian behavior in high shear stress ranges [5].

There are specific texts devoted to rheology [6,7] where we can find detailed descriptions of classical models. The simplest of them of quasi-Newtonian type is the Ostwald–de Waele or power-law model. More specific for example is the Walburn–Schneck model [8] for modeling blood flow, which includes among its parameters the amount of hematocrit H or fraction of red cells in the blood. The polymeric models consider other factors that modify the viscosity, such as the molecular weight, the polymeric concentration and the changes in the shape of the polymer chain [9]. Within the quasi-Newtonian classical models, the four parameter Carreau model and the five parameter Carreau–Yasuda model allows one to constrain the limit values in the viscosity when the power-law model predicts non-physical values at high and low shear stresses.

In this work we are interested in the finite element approximation of this type of problems. In particular, we aim to explore the possibilities and benefits of using a three-field formulation, having as unknowns the deviatoric stress, the velocity and the pressure. These mixed approximations can lead to different types of numerical instabilities inherent to the mathematical structure of the equations to be solved when the classical Galerkin approach is used. On the one hand, pressure and velocity are out of control unless appropriate inf–sup conditions are satisfied by the interpolation spaces. Conditions of this type need to be fulfilled also for the velocity and stress interpolation spaces. On the other hand, small viscosity values can lead to the classical instabilities found in convection dominated flows.

Referring to the compatibility conditions (see e.g. [10]), for the three-field approximation they consist of two restrictions, one between pressure and velocity and the other between velocity and stress [11]. These two restrictions reduce the choices of stable finite element spaces that allow one to discretize the unknowns. For example, in [12,13] it is shown how to design elements that satisfy the inf–sup condition between velocities and stresses through the addition of bubble functions. Another way to satisfy this restriction is using discontinuous finite element spaces for the stress, as shown in [14]. In the viscoelastic fluid context, a well-known stable interpolation in the two-dimensional case consists of using biquadratic elements for the velocity field, bilinear pressure interpolation and a multi-bilinear interpolation for the stresses, which is the popular Marchal–Crochet element [15]. The mathematical analysis of this element can be found in [16]. It is a clear example of the difficulties to satisfy the inf–sup conditions associated to the three-field formulation of flow problems.

When convection becomes dominant, it is necessary to use a stabilized formulation in any case. Among the methods that serve this purpose one can use the Streamline-Upwind/Petrov–Galerkin (SUPG) method [17], the Galerkin-Least Square (GLS) method [18], the Characteristics Galerkin method [19] or the Taylor–Galerkin method [20] (see [21]).

The two stabilized formulations proposed in this work are framed in the context of sub-grid scale (SGS) methods (also termed variational multi-scale methods) introduced by Hughes et al. [22] for the scalar convection–diffusion–reaction problem, and extended later to the vectorial Stokes problem in [23], where the space of the sub-grid scales is taken as orthogonal to the finite element space. The purpose of the present paper is to extend and test numerically the formulation presented in [11] for the Stokes problem with constant viscosity to a three-field formulation $\boldsymbol{\sigma}$–$\boldsymbol{u}$–p (deviatoric stress, velocity and pressure) of the Navier–Stokes problem with non-linear viscosity.

The starting point of a sub-grid scale approach is to split the unknowns of the problem into two components, namely, the component that can be approximated by the finite element mesh and the unresolvable one, called sub-grid scale or simply sub-scale in what follows. The latter needs to be approximated in a simple manner in terms of the former, so as to capture its main effect and yield a stable formulation for the finite element unknown. The number of degrees of freedom is therefore the same as for the Galerkin method. There are different ways to approximate the sub-scale and, in particular, to choose the (finite dimensional) space where it is taken. In this paper we will describe two formulations which precisely differ in this choice. In the first one, it will be equal to the space of finite element residuals (in a sense to be made precise in what follows), whereas in the second the space for the sub-scales will be taken as orthogonal to the finite element space. Both formulations will allow one to deal with the instabilities of the three-field formulation described earlier. There will be no need to meet the inf–sup conditions for the interpolation spaces and it will be possible to solve convection dominated problems.

We have performed a rather complete numerical testing of the formulations presented. The numerical results shown in this work can be separated into four groups. The first (Section 4.1) corresponds to the study of the convergence of