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Variational multi-scale stabilized formulations for the stationary three-field incompressible viscoelastic flow problem

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Abstract

In this paper, three-field finite element stabilized formulations are proposed for the numerical solution of incompressible viscoelastic flows. These methods allow one to use equal interpolation for the problem unknowns $\boldsymbol{\sigma}$–$\boldsymbol{u}$–p (elastic deviatoric stress–velocity–pressure) and to stabilize dominant convective terms. Starting from residual-based stabilized formulations, the proposed method introduces a term-by-term stabilization which is shown to have a superior behavior when there are stress singularities. A general discontinuity-capturing technique for the elastic stress component is also proposed, which allows one to eliminate the local oscillations that can appear when the Weissenberg number We is high and the fluid flow finds an abrupt change in the geometry. The formulations are tested in the classical 4:1 planar contraction benchmark up to We $= 5$ in the inertial case, with Reynolds number Re $= 1$, and up to We $= 6.5$ in the quasi non-inertial case, with Re $= 0.01$. The standard Oldroyd-B constitutive model is used for the rheological behavior and linear and quadratic elements for the spatial approximation.

Keywords: Viscoelastic fluids; Stabilized finite element methods; Oldroyd-B fluid; Discontinuity capturing technique

1. Introduction

The finite element approximation of the flow of viscoelastic fluids presents several numerical difficulties. It inherits obviously the problems associated with the approximation of the incompressible Navier–Stokes equations, mainly the compatibility between the velocity–pressure approximation and the treatment of the nonlinear advective term. But, on top of that, now the constitutive equation is highly nonlinear, with an advective term that may lead to both global and local oscillations in the numerical approximation. Moreover, even in the case of smooth solutions it is necessary to meet some additional compatibility conditions between the velocity and the stress interpolation in order to control velocity gradients. Elements that satisfy the compatibility requirements velocity–pressure and stress–velocity are rare.

The treatment of the nonlinearity is another aspect that deserves to be studied in detail. Apart from the nonlinearity in the convective term of the momentum equation, the constitutive equation has two additional nonlinear terms,

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namely, the convective one and the rotational one. Fixed point type schemes are robust, but with a very low convergence rate when the elastic component increases [1]. Newton–Raphson schemes are most extensively used in the literature [2,3], although they often need to be complemented with additional numerical tools, such as continuation methods and relaxation schemes. The results obtained in this work have been obtained using the Newton–Raphson method, which has produced good results in all the range of Weissenberg numbers analyzed ($0 \leq \mathrm{We} \leq 6.5$).

Once the equations have been properly linearized, the advective nature of the constitutive equation, which becomes dominant when the Weissenberg number increases, makes it necessary to use a stabilized finite element formulation to avoid global oscillations. The most widespread method to account for the convective term in the constitutive equation is the so-called SUPG method of Brooks and Hughes [4], first applied to viscoelastic flows by Marchal and Crochet [5]. In a more recent work, Masud et al. [2] use a Variational Multi-Scale (VMS) stabilized method for the momentum–continuity equations and the same SUPG method for the constitutive equation. Other stabilized methods for the viscoelastic fluid problem are the GLS-type methods used for example by Fan et al. [6] and Coronado et al. [7]. Different families of stabilized formulations can also be found in the literature. For example, Li et al. [8] proposed the so-called I–PS–DEVSS–CNBS scheme to stabilize the viscoelastic problem, based on the finite incremental calculus (FIC), pressure stabilization process, the discrete elastic–viscous stress-splitting method (DEVSS), the use of the Crank–Nicolson-based-splitting (CNBS) scheme, and the use of the non-consistent SU method to stabilize the viscoelastic equation. Other two options to circumvent the dominant convective nature of the problem are the fully explicit characteristic based split (CBS) scheme (see [9]) proposed by Nithiarasu [10], with a good performance for a wide range of Weissenberg numbers, and the nonlinear weighted least-squares finite element method proposed by Lee [1]. In the present work, we apply two stabilized formulations based on the VMS framework to control the convective nature of the viscoelastic constitutive equation, different to those just described.

The use of discontinuity-capturing (DC) techniques is not a popular topic in the analysis of viscoelastic flows, but the high elastic stress gradients that appear when the Weissenberg number is increased make it a typical situation where the application of a DC scheme can help. Carew et al. [11] have shown that the inclusion of such a DC technique in a stabilized formulation can improve the stability properties and permits to analyze fluids with a higher elasticity. In the work cited, the numerical diffusion of the discontinuity-capturing term is based on the finite element residual of the constitutive equation, in a similar way to that used by Codina [12] (see also [13]). In the present work we propose a numerical diffusion based on the orthogonal projection of the elastic stress gradient, which represents the *non-captured* part in the finite element approximation.

Referring to the compatibility conditions of inf–sup type for the viscoelastic three-field approximation, they consist of two restrictions on the interpolation spaces, one between pressure and velocity and the other between velocity and the elastic stress (see e.g. [14–16] for background). These two restrictions reduce drastically the choices of stable finite element spaces that allow one to discretize the unknowns. For example, in the paper of Marchal and Crochet [5] one can find different inf–sup stable elements capable to solve the viscoelastic problem. In this classical reference, the authors propose a family of biquadratic velocity and bilinear pressure elements with a multi-bilinear (2×2 or 3×3 or 4×4) stress element for the 2D case. The mathematical analysis of these elements can be found in [17]. It is a clear example of the difficulties to satisfy the two inf–sup conditions associated to the three-field formulation needed in the viscoelastic flow problem. For the tridimensional case, Bogaerds et al. [18] propose a DEVSS-DG stable spatial discretization using tri-quadratic interpolation for velocity, tri-linear interpolation for both pressure and discrete rate of deformation, while the viscoelastic stresses are approximated by discontinuous tri-linear polynomials. In [19] one can find a good review of mixed methods that satisfy the two compatibility conditions required.

The stabilized formulations proposed in this work have their roots in the context of VMS methods introduced by Hughes et al. [20] for the scalar convection–diffusion–reaction problem, and extended later to the vectorial Stokes problem in [21], where the space of the sub-grid scales is taken as orthogonal to the finite element space. As we shall see, this is an important ingredient in the design of our formulations. The purpose of the present paper is precisely to design and test numerically stabilized formulations for the viscoelastic fluid flow problem, permitting the use of equal interpolation between the unknowns (deviatoric elastic stress, velocity and pressure) even in cases where the elastic stress gradients and the elastic component of the fluid are important.

The starting point of a VMS approach is to split the unknowns of the problem into two components, namely, the component that can be approximated by the finite element mesh and the unresolvable one, called sub-grid scale or simply sub-scale in what follows. The latter needs to be approximated in a simple manner in terms of the former, so as to capture its main effect and yield a stable formulation for the finite element unknown. The number of degrees of