Comput. Methods Appl. Mech. Engrg. 274 (2014) 1–18

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Stabilized reduced basis method for parametrized advection–diffusion PDEs

Paolo Pacciarini [a], Gianluigi Rozza [b,c,*]

[a] MOX – Modellistica e Calcolo Scientifico, Dipartimento di Matematica F. Brioschi, Politecnico di Milano, P.za Leonardo da Vinci 32, I-20133 Milano, Italy
[b] SISSA mathLab, International School for Advanced Studies, Via Bonomea 265, I-34136 Trieste, Italy
[c] CMCS – Modelling and Scientific Computing, MATHICSE – Mathematics Institute of Computational Science and Engineering, EPFL – Ecole Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 9 August 2013
Received in revised form 24 November 2013
Accepted 3 February 2014
Available online 12 February 2014

Keywords:
Reduced basis
Advection dominated problems
Stabilization methods

ABSTRACT

In this work, we propose viable and efficient strategies for the stabilization of the reduced basis approximation of an advection dominated problem. In particular, we investigate the combination of a classic stabilization method (SUPG) with the Offline–Online structure of the RB method. We explain why the stabilization is needed in both stages and we identify, analytically and numerically, which are the drawbacks of a stabilization performed only during the construction of the reduced basis (i.e. only in the Offline stage). We carry out numerical tests to assess the performances of the "double" stabilization both in steady and unsteady problems, also related to heat transfer phenomena.

1. Introduction

The aim of this work is to study and develop a *stabilized reduced basis method* suitable for the approximation of the solution of parametrized advection–diffusion PDEs with high Péclet number, that is, roughly, the ratio between the advection term and the diffusion one.

Advection–diffusion equations are very important in many engineering applications, because they are used to model, for example, heat transfer phenomena (with conduction and convection) [16] or the diffusion of pollutants in the atmosphere or in the water [6,26]. In such applications, we often need very fast evaluations of the approximated solution, depending on some physical and/or geometrical input parameters. This happens, for example, in the case of *real-time* simulations. Moreover, we need rapid evaluations also if we have to perform repeated approximation of the solution, for different input parameters. An important case of this *many-query* situation is represented by some optimization problems, in which the objective function to optimize depends on the parameters through the solution of a PDE.

The reduced basis (RB) method [25,29] meets our need for rapidity and it is also able to guarantee the *reliability* of the solution, thanks to sharp *a posteriori* error bounds. A crucial feature of the RB method is its decomposition into two computational steps. During the first expensive one, called *Offline* step, some high-fidelity approximated solutions are computed, which will become the global basis functions of the Galerkin projections performed during the second inexpensive phase, called *Online* step. A brief introduction to the RB method will be given in Section 2.

* Corresponding author at: SISSA mathLab, International School for Advanced Studies, Via Bonomea 265, I-34136 Trieste, Italy. Tel.: +39 0403787451; fax: +39 0403787528.

E-mail addresses: paolo.pacciarini@polimi.it (P. Pacciarini), gianluigi.rozza@sissa.it (G. Rozza).

http://dx.doi.org/10.1016/j.cma.2014.02.005
0045-7825/

As the advection–diffusion equations are often used to model heat transfer phenomena, we can find in literature many results about the RB approximation of heat transfer problems such as the Poiseuille–Graetz problem or the "thermal fin" problem [9,20,25,28,30,31]. However, until now, only the case in which the Péclet number is reasonably low (i.e. $\sim 10^2$) was considered without stabilization.

When the Péclet number is higher (i.e. $\sim 10^5$), it is very well known [27] that the Finite Element (FE) solution of the advection–diffusion equation - that the RB method aims to recover - can show significant instability phenomena. In order to fix this problem, in the RB framework, some solutions have been proposed for the steady case [6,7,24,26]. The basic idea is to consider as *truth* solution a stabilized FE one, using some classical stabilization method (e.g. the SUPG method [27]), and then to perform the RB *Offline* and *Online* steps using the stabilized bilinear form instead of the original one. In the cited papers we can find some applications to environmental sciences and engineering problems concerning, in particular, air pollution. Very recently, also a Petrov–Galerkin based strategy has been proposed to deal with high Péclet number problems [5].

In some of the previous works, the issue of stability was not studied too much inside concerning the Offline–Online affine decomposition and some proposed options were not very viable for more complex problems [26]. Our aim was not only to increase the Péclet number in parametrized problems dealing with convection and transport, including parametrized moving inner fronts and boundary layers. This work has been also motivated by the aim of creating a general stabilized framework to provide some explanations of previous approaches and known results in literature.

In our work we then want to go further in the study of the *stabilized RB method*, proposing viable and efficient strategies to be used combined with the Offline–Online computational procedures and providing a deeper analysis on the need of stabilization for parametrized advection–diffusion problems. We start by studying steady problems and then we move to the time dependent case.

After having done, in Section 2, a short presentation of the RB method, in Section 3 we observe and analyse what happens when we "stabilize" only the Offline stage of the RB method, thus producing "stable" basis functions to be interpolated in the Online stage by projecting with respect to the non-stabilized advection–diffusion operator. We will show that, in general, the latter strategy is not satisfactory because of "inconsistency" problems between the Offline and Online stages, arising from the use of two different bilinear forms. We will also prove an *a priori* error estimate (Proposition 3.1) in order to estimate this inconsistency. After having determined which stabilization strategy gives better results and why, in Section 4 we will try to apply it to a test problem with a parameter dependent internal layer, using also a piecewise quadratic polynomial *truth* approximation space. Finally, in Section 5 we extend the investigation of the RB stabilization method to parabolic problems.

2. A brief review of the reduced basis method

The reduced basis (RB) method is a reduced order modelling (ROM) technique which provides rapid and reliable solutions for parametrized partial differential equations (PPDEs), in which the parameters can be either physical or geometrical [25,29].

The need to solve this kind of problems arises in many engineering applications, in which the evaluation of some *output* quantities is required. These *outputs* are often function of the solution of a PDE, which can in turn depend on some *input* parameters. The aim of the RB method is to provide a very fast computation of this *input–output* evaluation.

Roughly speaking, given a value of the parameter, the (Lagrange) RB method consists in a Galerkin projection of the continuous solution on a particular subspace of a high-fidelity approximation space, e.g. a finite element (FE) space with a large number of degrees of freedom. This subspace is the one spanned by some pre-computed high-fidelity global solutions (*snapshots*) of the continuous parametrized problem, corresponding to some properly chosen values of the parameter.

For a complete presentation of the reduced basis method we refer to [25,29], now we just recall its main features and we introduce some notations.

2.1. The continuous problem

Let $\boldsymbol{\mu}$ belong to the *parameter domain* $\mathcal{D}$, a subset of $\mathbb{R}^P$. Let Ω be a regular bounded open subset of $\mathbb{R}^d$, $(d = 1, 2, 3)$ and X a suitable Hilbert space. Given a parameter value $\boldsymbol{\mu} \in \mathcal{D}$, let $a(\cdot, \cdot; \boldsymbol{\mu}) : X \times X \to \mathbb{R}$ be a bilinear form and let $F(\cdot; \boldsymbol{\mu}) : X \to \mathbb{R}$ be a linear functional. As we will focus on advection–diffusion equations, that are second order elliptic PDE, the space X will be such that $H_0^1(\Omega) \subset X \subset H^1(\Omega)$. Formally, our problem can be written as follows:

$$\begin{aligned} &\text{find } u(\boldsymbol{\mu}) \in X \text{s.t.} \\ &a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = F(v; \boldsymbol{\mu}) \quad \forall v \in X. \end{aligned} \tag{1}$$

The coercivity and continuity assumption on the form a can now be expressed by, respectively:

$$\exists\ \alpha_0 > 0 \quad \text{s.t.} \quad \alpha_0 \leqslant \alpha(\boldsymbol{\mu}) = \inf_{v \in X} \frac{a(v, v; \boldsymbol{\mu})}{\|v\|_X^2} \quad \forall \boldsymbol{\mu} \in \mathcal{D} \tag{2}$$

and

$$+\infty > \gamma(\boldsymbol{\mu}) = \sup_{v \in X} \sup_{w \in X} \frac{|a(v, w; \boldsymbol{\mu})|}{\|v\|_X \|w\|_X} \quad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{3}$$