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Partitioned treatment of uncertainty in coupled domain problems: A separated representation approach

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ABSTRACT

This work is concerned with the propagation of uncertainty across coupled domain problems with high-dimensional random inputs. A stochastic model reduction approach based on low-rank separated representations is proposed for the partitioned treatment of the uncertainty space. The construction of the coupled domain solution is achieved though a sequence of approximations with respect to the dimensionality of the random inputs associated with each individual sub-domain and not the combined dimensionality, hence drastically reducing the overall computational cost. The coupling between the sub-domain solutions is done via the classical finite element tearing and interconnecting (FETI) method, thus providing a well suited framework for parallel computing. Two high-dimensional stochastic problems, a 2D elliptic PDE with random diffusion coefficient and a stochastic linear elasticity problem, have been considered to study the performance and accuracy of the proposed stochastic coupling approach.

1. Introduction

Simulation-based prediction of most physical systems is subject to either lack of knowledge about the governing physical laws or incomplete/limited information about model parameters such as material properties, initial, or boundary conditions. In order to obtain realistic predictions of these systems one, therefore, needs to characterize such uncertainties and quantify their impact on quantities of interest (QoI). Uncertainty quantification (UQ), an emerging field in computational engineering and science, is concerned with the development of rigorous and efficient solutions to this exercise.

A major class of UQ approaches are probabilistic, wherein uncertain parameters are represented by random variables or processes. Among these techniques, stochastic spectral methods based on polynomial chaos (PC) expansions [27,60] have received special attention due to their advantages over traditional perturbation-based and Monte Carlo sampling (MCS) approaches. In particular, under certain regularity conditions, these schemes converge faster than MCS methods and, unlike perturbation methods, are not restricted to problems with small uncertainty levels [27]. Stochastic spectral methods are based on expanding the solution of interest in PC bases. The coefficients of these expansions are then computed, for instance, via Galerkin projection [27], referred to as the stochastic Galerkin (SG), or pseudo-spectral collocation [42,59], named stochastic collocation (SC). Although PC-based techniques benefit from elegant mathematical analyses, e.g., formal convergence

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studies, they may suffer from the so-called *curse-of-dimensionality* if executed carelessly: the computational cost may grow exponentially as a function of the number of independent random inputs [58,41,17].

Of particular interest in the present study is the problem of uncertainty propagation across coupled domain problems where independent, high-dimensional random inputs are present in each sub-domain. In such cases, the solution depends on all random inputs; hence, a direct application of PC expansion may not be feasible or at least not desirable. While integration of PC expansions with standard domain decomposition (DD) techniques may partially reduce the overall computational complexity by partitioning the physical space, e.g., see [28,13,54], expansions (or sampling) with respect to the combined set of random inputs is still required. Instead, we propose an approach that additionally enables a *partitioned treatment* of the stochastic space; that is, the solution is computed through a sequence of approximations with respect to the random inputs associated with each individual sub-domain, and not the combined set of random inputs. To this end, we adopt a stochastic expansion based on the so-called *separated representations* and demonstrate how it can be obtained in conjunction with a DD approach. As one of the simplest cases of such coupling, we here consider a linear problem on two non-overlapping sub-domains with a common interface. We later provide a numerical experiment featuring three coupled sub-domains. Although more elaborate coupling techniques are possible, we restrict ourselves to a finite element tearing and interconnecting (FETI) approach [22] for the sake of simplicity.

Separated representation, also known as canonical decomposition or parallel factor analysis, was first introduced in [31] to represent a multi-way tensor by a finite sum of rank-one tensors. Consequently, it has been extensively applied to several areas including image compression and classification [50,24], telecommunication [51,38], chemometrics [9], data mining [36,34,56,2], solution of operator equations [37,5,6,3], among other applications. For a comprehensive review of separated representations and their applications we refer the interested reader to [35]. Approximation techniques based on separated representation of high-dimensional stochastic functions have also been recently proposed in [43,16,15,44,45,32,30,18] and proven effective in reducing the issue of curse-of-dimensionality. We here adopt a special form of separated representations for the stochastic computation of coupled domain problems.

Let $(\Omega, \mathcal{T}, \mathcal{P})$ be a complete probability space where Ω is the sample set and $\mathcal{P}$ is a probability measure on the σ-field $\mathcal{T}$. Assume that the input uncertainty has been discretized and approximated by random variables, such that the vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) : \Omega \to \mathbb{R}^d,\ d \in \mathbb{N}$, represents the set of independent random inputs associated with a PDE defined on a domain $\mathcal{D} \subset \mathbb{R}^D,\ D \in \{1, 2, 3\}$. As displayed in Fig. 1, we consider the case in which $\mathcal{D}$ is composed of two non-overlapping sub-domains $\mathcal{D}_1$ and $\mathcal{D}_2$ ($\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$) sharing an interface boundary $\Gamma_I = \partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$. We further assume that the random inputs representing the uncertainty associated with $\mathcal{D}_1$ and $\mathcal{D}_2$ are independent and denoted by $\boldsymbol{\xi}_1 = (\xi_{1,1}, \ldots, \xi_{1,d_1}) : \Omega \to \mathbb{R}^{d_1}$, and $\boldsymbol{\xi}_2 = (\xi_{2,1}, \ldots, \xi_{2,d_2}) : \Omega \to \mathbb{R}^{d_2}$, respectively. Here, $d_1, d_2 \in \mathbb{N}$ are the sizes of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, respectively, and $d = d_1 + d_2$. Letting $u_1(\boldsymbol{x}, \boldsymbol{\xi}) : \bar{\mathcal{D}}_{\mathbf{1}} \times \Omega \to \mathbb{R}$ and $u_2(\boldsymbol{x}, \boldsymbol{\xi}) : \bar{\mathcal{D}}_{\mathbf{2}} \times \Omega \to \mathbb{R}$ be the sub-domain solutions, we consider the separated representation of the form

$$u_i(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{l=1}^{r} u_{0,i}^l(\boldsymbol{x}) \phi_1^l(\boldsymbol{\xi}_1) \phi_2^l(\boldsymbol{\xi}_2) + \mathcal{O}(\epsilon), \quad i = 1, 2. \tag{1}$$

Here $u_{0,i}^l(\boldsymbol{x}) : \bar{\mathcal{D}}_i \to \mathbb{R}$ and $\phi_i^l(\boldsymbol{\xi}_i) : \Omega \to \mathbb{R},\ l = 1, \ldots, r$, are, respectively, deterministic and stochastic functions – or *factors* – to be determined along with the *separation rank* r. These quantities are not fixed *a priori* and are computed through an optimization scheme such that a prescribed target accuracy ϵ is reached for a minimum r.

We note that the expansion in (1) is constructed by the multiplication of the unknown factors $u_{0,i}^l(\boldsymbol{x}),\ \phi_1^l(\boldsymbol{\xi}_1)$, and $\phi_2^l(\boldsymbol{\xi}_2)$; hence, finding (1) may be cast in the form of a non-linear problem. However, as we shall describe later, (1) may be computed through a sequence of alternating linear problems, where $\{u_{0,i}^l(\boldsymbol{x})\}_{l=1}^r,\ i = 1, 2,\ \{\phi_1^l(\boldsymbol{\xi}_1)\}_{l=1}^r$, or $\{\phi_2^l(\boldsymbol{\xi}_2)\}_{l=1}^r$ are solved for one at a time while others are fixed at their recent values. This alternating construction, together with the separated form of (1), enables computing the stochastic functions $\{\phi_1^l(\boldsymbol{\xi}_1)\}_{l=1}^r$ and $\{\phi_2^l(\boldsymbol{\xi}_2)\}_{l=1}^r$ with computational complexities that depend on d_1 and d_2 but not $d = d_1 + d_2$. This will be delineated further in Sections 3.2 and 4.4. Additionally, the representation (1) allows for a natural extension of DD techniques for computing $\{u_{0,i}^l(\boldsymbol{x})\}_{l=1}^r$. In the present work, we employ the standard FETI approach [22] for this purpose. Moreover, for situations where the separation rank r is small, (1) provides a reduced order approximation and representation of the coupled solution. We will describe the details of computing (1) in Section 4.

Among the limited earlier effort on separating random inputs for the solution of coupled problems, we particularly mention the recent work of Arnst et al. [4] that is based on the so-called *reduced chaos* expansions [53]. In particular, in each component, e.g., physics or sub-domain, the solution is expanded in a PC basis that is generated based on random inputs

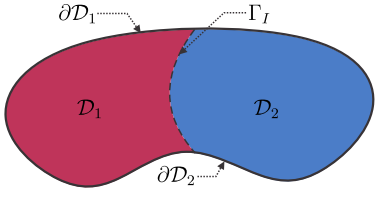

Fig. 1. Geometry of the original problem and partitioning of $\mathcal{D}$ into two non-overlapping sub-domains $\mathcal{D}_1$ and $\mathcal{D}_2$. $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2,\ \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, and $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2 = \Gamma_I$.