Comput. Methods Appl. Mech. Engrg. 274 (2014) 237–263

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Improving the *k-compressibility* of Hyper Reduced Order Models with moving sources: Applications to welding and phase change problems

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ARTICLE INFO

Article history:
Received 27 September 2013
Received in revised form 25 January 2014
Accepted 17 February 2014
Available online 3 March 2014

Keywords:
Reduced Order Models
Hyper reduction
Welding
Phase change
Moving sources
Proper Orthogonal Decomposition

ABSTRACT

The simulation of engineering problems is quite often a complex task that can be time consuming. In this context, the use of Hyper Reduced Order Models (HROMs) is a promising alternative for real-time simulations. In this work, we study the design of HROMs for non-linear problems with a moving source. Applications to nonlinear phase change problems with temperature dependent thermophysical properties are particularly considered; however, the techniques developed can be applied in other fields as well.

A basic assumption in the design of HROMs is that the quantities that will be hyper-reduced are *k-compressible* in a certain basis in the sense that these quantities have at most k non-zero significant entries when expressed in terms of that basis. To reach the computational speed required for a real-time application, k must be small. This work examines different strategies for addressing hyper-reduction of the nonlinear terms with the objective of obtaining *k-compressible* signals with a notably small k. To improve performance and robustness, it is proposed that the different contributing terms to the residual are separately hyper-reduced. Additionally, the use of moving reference frames is proposed to simulate and hyper-reduce cases that contain moving heat sources. Two application examples are presented: the solidification of a cube in which no heat source is present and the welding of a tube in which the problem posed by a moving heat source is analysed.

1. Introduction

Modelling problems that are intractable from the point of view of reaching real time computational speed are quite frequently found in science and in engineering. Two particularly time-consuming cases are the problems of welding and nonlinear phase change. This paper presents hyper-reduction methods for these two problems, but the methods developed can be extended to many other problems with similar characteristics.

A welding problem is essentially a thermally driven process. Due to this fact, a correct description of the heat source that represents the energy input is of great importance. Generally, this input is described in terms of a standardised and highly concentrated moving heat source. This feature sets up a problem whose main characteristics are rather rapid changes in the

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http://dx.doi.org/10.1016/j.cma.2014.02.011
0045-7825/

involved fields as well as the rapid variation of material properties. The concentrated behaviour of the heat source introduces constraints in the size of the mesh and time step used to run the simulation, thus forcing an increment in the number of degrees of freedom of the problem.

Simulation of a welding problem is a highly complex task described as a Thermo-Mechanical-Metallurgical (TMM) process [1]. To reduce this complexity, certain simplifications are applied, such as describing the governing physics of the problem with a staggered thermo-mechanical model [2]. Despite efforts to render treatment of the problem more amenable, its complexity makes it unaffordable from the perspective of real-time simulations. In this context, the design of Reduced Order Models (ROMs) is an elegant and promising alternative to the classical high fidelity solutions.

Currently, the use of separate representations to build ROMs has caught the attention of the engineering community. The manner in which the separated representation is built is described by two approaches [3]. In one approach, *a posteriori* model reduction techniques require knowledge of the solution to a training problem. The most prominent *a posteriori* technique is based on the Proper Orthogonal Decomposition (POD) method [4,5]. For the second approach, the *a priori* model reduction techniques require no previous knowledge of the solution, concept which was introduced by Ryckelynck in [6]. In this context, the leading technique is the Proper Generalised Decomposition (PGD), which has its roots in the works of Ladevèze [7,8].

In both approaches, to successfully address the high dimensionality of the problem, it is assumed that the solution can be described in terms of a reduced number of functions in the separate representation. In a certain sense, this situation can be referred to as the *separate representation hypothesis*. A consequence of this idea is the supposition that the solution of the problem is *k-compressible* in a certain basis, with a notably small k. The solution is said to be *k-compressible* if it has at most k non-zero significant entries when expressed in terms of that *magic* basis. An extensive analysis of similar concepts is offered in the context of compressed sensing, see, for example [9,10].

The present work considers the *a posteriori* ROM technique based on the POD method. The reduction of the problem begins by reducing the dimensionality of the discrete versions of the test and trial spaces. This process is generally carried out by finding a basis, say $\boldsymbol{\Theta}$, in which the solution to the problem is *k-compressible*, which is obtained by computing the Singular Value Decomposition of a set of snapshots of the solution. Next, the trial solution space is defined as an affine translation of $span\{\boldsymbol{\Theta}\}$. If a Bubnov–Galerkin projection is used, the test functions are in $span\{\boldsymbol{\Theta}\}$. It must be mentioned that a Petrov–Galerkin projection is recommended for problems where the Jacobian is not symmetric positive definite (SPD), which involves the solution of a least squares problem [11].

After reducing the dimensionality of the test and trial spaces, the size of the system of linear equations to be solved is reduced from a size of $N \times N$, where N is the size of the high fidelity (HF) model, to a size of $k \times k$, with $k \ll N$. With this approach, although the computational cost of solving the system of linear equations is reduced, the cost of assembling the residual and the tangent matrix at each Newton iteration is still of order N. It is widely known that to significantly reduce the computational complexity of the problem, the cost of assembling the residual and the tangent matrix at each Newton iteration must be reduced [11–16]. To accomplish this objective, a second reduction is performed by evaluating the involved quantities at a few points of the domain. Extensions of this idea have been used in the context of *a priori* and *a posteriori* reduction methodologies. For instance, in [17] the extension to nonlinear Finite Element models making use of an *a priori* approach is presented and the term hyper-reduction is coined to refer to the general procedure of performing a second reduction. Another work following this line is that of Sarbandi et al. [18].

In the context of *a posteriori* reduction techniques, the hyper-reduction method has been widely applied. Generally, the ideas along this path are based on the gappy data reconstruction method introduced by Everson and Sirovich [19] in the image processing community. For example, an extension of this idea to Finite Volume equations was accomplished by Astrid [20], extensions to nonlinear mechanical models are found in [13,21] and, in the case of computational fluid dynamics, treatments of this kind are found in [14,15]. In the present work, we use the term Hyper-Reduced Order Models (HROMs) to refer to the reduced models arising from the hyper-reduction method. To the authors' best knowledge, the design and application of HROMs specifically suited for phase change and welding problems have not yet been addressed in the literature.

In this paper, different approaches are studied in detail for the design of HROMs with particular application to the nonlinear phase change problem. Schemes in which the residual is hyper-reduced as a unit, taking the history of the residual as snapshots for the gappy data reconstruction procedure, are usually found in the literature [11,14]. This strategy is used as a reference technique for comparison in this work. As observed from numerical experiments, poor *k-compressibility* and tangent matrix conditioning are obtained when applying this technique for the design of HROMs. To improve the performance and robustness, it is proposed that the different contributing terms to the residual are separately hyper-reduced. These terms are assumed to be physically based nonlinear generalised contributing forces, features that lead to a well-posed HROM. In the case of welding problems, the moving heat source represents an issue that can severely affect the *k-compressibility* of the involved terms. This complication is addressed by considering both moving and fixed frames of reference respect to the welded piece.

The paper is organised as follows. Section 2 states the mathematical formulation of the solid–liquid phase change problem, and Section 3 presents the formulation of Reduced Order Models. The cost of assembling the nonlinear forces and tangent matrices is reduced by means of Hyper Reduced Order Models in Section 4, significantly reducing the computational complexity of the problem. In Section 5, the issue presented by the moving heat source is considered, and two application examples are presented in Section 6 to assess the performance of the introduced HROMs. Specifically, the solidification of a cube and welding of a tube without material deposition are analysed. Finally, Section 7 describes the main conclusions of this work.