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A priori space–time separated representation for the reduced order modeling of low Reynolds number flows

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ABSTRACT

Reduced order models (ROMs) in fluid dynamics are nowadays mostly developed by performing a projection of the Navier–Stokes equations onto a low-dimensional space basis. This basis is usually obtained through Proper Orthogonal Decomposition (POD), which remains one the most efficient techniques to compress precomputed data. The main drawback of *a posteriori* POD based ROMs is however their lacks of reliability as their parameters are varied, preventing their direct uses within optimization algorithms. The goal of the present article is to obtain an *a priori* low-dimensional space–time separated representation of the fluid fields, without precomputed data. The approach is based on the use of space–time Proper Generalized Decomposition (PGD) definitions, which are successfully applied in several fields but whose uses in fluid dynamics remain scarce. Their applications to the Navier–Stokes equations are indeed not straightforward, due to the pressure–velocity coupling, the divergence-free constraint and the non-linear convective term. The ROMs are built here from a space–time weak formulation of the Chorin–Temam prediction-correction scheme. More particularly, *a priori* space–time separated representations are obtained by applying the Galerkin based progressive PGD definition. The minimax PGD definition is moreover experimented on a linear Stokes simplified case. The related algorithms are explicitly given and illustrated on a transient lid-driven cavity flow. It is shown that the *a priori* space–time separated representations converge to the full model solution as the decomposition order increases. The ability of the resulting ROM to learn iteratively from its own error is highlighted: the progressive PGD algorithm can be used to effectively enrich incomplete POD and PGD precomputed space bases, such as those obtained with different parameter values. This may allow to ensure the accuracy of a ROM as the parameter is varied, which is of crucial interest for optimization problems.

1. Introduction

Reduced order models (ROMs) are nowadays able to represent complex systems with few degrees of freedom and at the cost of a moderate loss of accuracy. Numerous techniques can be found in the literature to build a ROM. Most of them involve the knowledge of a low-dimensional basis as starting point, such as Proper Orthogonal Decomposition (POD) [1,2], balanced POD [3,4], modal basis [5,6], balanced truncation [7,8], Krylov subspaces [9] and Centroidal Voronoi Tesselation [10], just to

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name a few. For incompressible fluid flows, the most popular method is probably the POD: due to both its optimal energetic convergence property and its applicability to non-linear problems, it is proven to be an efficient technique to obtain an *a posteriori* very low-dimensional space basis. *A posteriori* means here that the approach requires a precomputed quantity of data, more precisely snapshots of the velocity field, and optionally the pressure one, at different time steps, to compute the low-dimensional subspace. POD based ROMs are then classically obtained by performing a Galerkin projection, or a Petrov–Galerkin one, of the Navier–Stokes equations onto the POD space eigenfunctions [11–20]. The resulting ROMs may finally be used within optimization algorithms, with interesting prospects for the control of fluid flows [21–25].

One of the well-known drawbacks of POD based ROMs is their lacks of accuracy and reliability as their internal parameters are varied, which is the case in optimization or parametric studies. Without special care, they are indeed limited to rapidly reproduce the full model numerical solution, which has already been computed to obtain the reduced basis at a reference value of the parameter. At best, they are useful for long-time predictions and to obtain an estimate of the fluid flow in a very close vicinity of the parameter reference value. Roughly speaking, if the information is not contained in the precomputed data, then the POD based ROM is not able to simulate it accurately. Both theoretical and computational efforts have been performed to enhance its accuracy as parameters are varied, for instance and not exhaustively:

- A line of research focuses on the interpolation of POD bases precomputed at different values of the parameters, to yield a space basis adapted to a new value of the parameter. More precisely, the subspace angle interpolation [26,27], as well as an advanced interpolation based on the Grassmann manifold and its tangent space [28], are developed and successfully tested in the field of aeroelasticity: new orthogonal space functions can be rapidly computed and used to build an adapted ROM. Since these approaches are essentially geometrical and do not take into account the underlying equations, the precomputed bases still require to be close enough to the new parameter value to yield an accurate ROM. Moreover, their applications to incompressible fluid flows may not be straightforward, due to the additional divergence-free constraint on the velocity field: the interpolated space functions are not themselves guaranteed to be divergence-free.
- A second approach is based on the trust-region POD technique [23,29]: a ROM is used within the optimization loop, and its range of validity is automatically estimated with regard to the optimization goal by using the trust-region method. It can therefore be updated only when it is required, which enables to save a lot of CPU time compared to the direct use of the full model within the loop. Nevertheless, the update step of the ROM is ultimately achieved with the full model computation to build a new POD space basis.

Another technique consists in progressively enriching the ROM by the equation residual and the related Krylov subspaces. The approach, called A Priori Hyper-Reduction [30,31], has been successfully developed on linear and non-linear model problems, for instance on transfer and Burgers equations [32–34]. The first try on the Navier–Stokes equations [35] shows that the equations residual is inadequate to built a ROM from scratch, starting only from initial fields, and to advance forward in time by iterative enrichment. A second try [36] proves that the equations residual can be used to efficiently stabilize a POD based ROM. It is also shown that the residual is not able to enrich the ROM when a parameter is varied.

In other fields, alternative numerical approaches are now available to build a low-dimensional separated representation of solutions, without knowledge of precomputed data. More precisely, *a priori* space–time separated representation has first been proposed by Ladevèze, under the name of radial approximation, with applications to nonlinear structural mechanics [37]. This technique of separation of variables, called nowadays Proper Generalized Decomposition (PGD), is receiving a growing interest and may be seen as a generalization of the POD for the *a priori* construction of a separated representation of the solution. It is applied in numerous fields [38,39], from the kinetic theory modeling of complex fluids [40,41] to the biology [42] and quantum chemistry [43]. It is moreover proved to be efficient for multidimensional problems, for instance on parametric deterministic heat transfer and conduction equations [44,45], on parametric models in evolving domains [46], and developed along with the LATIN method [47,48] to explore rationally the space of parameters [49,50]. It is also developed and successfully used in the stochastic framework [51–53]. In references [54–57], space PGD techniques are applied on the Navier–Stokes equations. More precisely, each velocity component at time t^n is searched under the form $u(x,y,t^n) \approx \sum_{i=1}^{m} \alpha_i X_i(x) Y_i(y)$, and new space functions for each direction have to be recomputed at each time step. The approach is shown to be efficient from the CPU time point of view. It is nevertheless mainly limited to simple geometries, i.e. to a separated fluid domain $\Omega = \Omega_x \times \Omega_y$, even if it may be generalized to arbitrary Ω by embedding it into a larger separated domain [58]. Furthermore, since the space functions are different at each time step, the resulting decomposition generates no space basis on a whole time interval, preventing the building of a ROM and a potential use within optimization loops.

While space–time PGD algorithms have been proven to be efficient to obtain *a priori* space–time separated representations in several fields, their applications to the Navier–Stokes equations are not straightforward, due to the pressure–velocity coupling, the divergence-free constraint and the non-linear convective term. The first successful attempt to apply a space–time PGD definition for low-Reynolds numbers incompressible flows is achieved in the recent article of Aghighi et al. [59]. A penalty formulation of the incompressibility constraint is used and the pressure is therefore totally removed in the momentum equation. The approach is proven to be efficient from the CPU time point of view compared to the full model. With this method, the resulted velocity field is not truly divergence-free and the pressure field does not have a physical meaning. We propose here an alternative way to face the velocity–pressure coupling and the divergence-free constraint, by applying PGD algorithms on a Chorin–Temam predictor–corrector scheme [60,61]. This enables to obtain, by construction, a