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Dual-primal domain decomposition method for uncertainty quantification

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ABSTRACT

The spectral stochastic finite element method (SSFEM) may offer an efficient alternative to the traditional Monte Carlo simulations (MCS) for uncertainty quantification of large-scale numerical simulations. In the framework of the intrusive SSFEM, the main computational challenge involves solving a coupled set of deterministic linear systems. For large-scale numerical models, the computational efficiency of the intrusive SSFEM primarily depends on the solution techniques employed to tackle the resulting coupled linear systems. In this paper, we report a probabilistic version of the dual-primal domain decomposition method for the intrusive SSFEM in order to exploit high performance computing platforms for uncertainty quantification. In particular, we formulate a probabilistic version of the dual-primal finite element tearing and interconnect (FETI-DP) technique to solve the large-scale linear systems in the intrusive SSFEM. In the probabilistic setting, the operator of the dual interface system in the dual-primal approach contains a coarse problem. The introduction of the coarse problem in the probabilistic setting leads to a scalable performance of the dual-primal iterative substructuring method for uncertainty quantification of large-scale computational models. The convergence properties, numerical and parallel scalabilities of the probabilistic FETI-DP method and the recently developed probabilistic version of the balancing domain decomposition by constraints (BDDC) method are contrasted. For numerical illustrations, we consider flow through porous media and linear elasticity problems with spatially varying system parameters modelled as non-Gaussian random processes. The algorithms are implemented on a Linux cluster using MPI and PETSc parallel libraries.

1. Introduction

Advances in high performance computing systems and parallel algorithms enable extreme scale computational simulations using high resolution numerical models. To establish credibility in these numerical predictions, it is imperative to quantify uncertainty in such large-scale simulations. The computational cost of the traditional Monte Carlo simulations (MCS) becomes intensive for large-scale models. To this end, the spectral stochastic finite element method (SSFEM) received considerable attention as an efficient alternative to MCS (e.g. [3–10]). In the so-called intrusive SSFEM based on the Galerkin projection, the computational efficacy of this approach is dictated by the solution technique adopted to tackle a set of coupled deterministic linear systems.[1]

To reduce the computational cost of the intrusive SSFEM approach, the reduced orthonormal vector basis technique [11] and stochastic reduced basis method [12–14] have been proposed in the literature. In this paper, we focus our attention on the development of a scalable domain decomposition solver for the intrusive SSFEM in order to effectively exploit high performance computing platforms for uncertainty quantification of large-scale numerical models.

In the intrusive SSFEM, the polynomial chaos expansion (PCE) is used to represent the uncertain input and output processes. The chaos coefficients of the output process are obtained by the Galerkin projection technique [3–10,15,16]. For high resolution numerical models, this methodology necessitates the solution of a large-scale deterministic linear system for the chaos coefficients of the output process. To effectively exploit the high performance computing platforms, specialized solution strategies are required to tackle this system.

The development of efficient solution techniques for the SSFEM linear system has received a considerable attention in the literature (e.g. [17–25]). The previous initiatives are focused towards developing efficient and robust preconditioners for the iterative solution techniques of the SSFEM linear system (e.g. [26,27]). To effectively exploit the high performance computing platforms, the preconditioners for the SSFEM must demonstrate a scalable

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performance to large number of processors. For parallel efficiency, the convergence rate of the iterative solvers should, for instance, not only be independent of the problem size and number of subdomains, but also be insensitive to the level of uncertainty and order of the stochastic dimensions.

To this end, a mathematical framework for non-overlapping domain decomposition method of SPDEs is introduced in [28] to quantify uncertainty in large-scale numerical simulations. In particular, non-overlapping domain decomposition method or substructuring technique is used to decompose the physical domain while the PCE based functional expansion is employed along the stochastic dimension. This approach is extended by the authors [29–31] to develop iterative substructuring technique for SPDEs. Specifically, a number of parallel one-level domain decomposition preconditioners (namely Lumped, Weighted Lumped and Neumann–Neumann preconditioners) are formulated for the iterative solution of the extended Schur complement system in the framework of SSFEM. The one-level preconditioners demonstrate a reasonable performance for moderate range of subdomains [29–31]. To enhance the performance further, a two-level domain decomposition preconditioner for the stochastic system is proposed by the authors [1,2,32,33]. For SPDEs, the algorithm may be considered as a probabilistic extension of the balancing domain decomposition by constraints (BDDC) [34,35].

For deterministic PDEs, the dual-primal finite element tearing and interconnect (FETI-DP) domain decomposition solver is shown to be scalable with respect to mesh size, subdomain size and fixed problem size per subdomain [36,37]. The condition number of the preconditioned FETI-DP operator is shown to be bounded as the number of subdomains with fixed problem size increases [38]. In this investigation, the research initiative is directed towards developing a probabilistic FETI-DP solver that is scalable not only with respect to the geometric parameters, but also with respect to the strength of randomness and the order of the stochastic expansion.

For deterministic PDEs, BDDC [35,39,40] and FETI-DP [36,37] are perhaps the most popular non-overlapping domain decomposition techniques for the iterative solution of large-scale deterministic linear systems. BDDC provides a two-level preconditioner for the solution of the so-called *primal interface problem* (Schur complement system). On the contrary, FETI-DP iteratively solves a system of Lagrange multipliers for the so-called *dual interface system*. While the coarse problem essential for scalability is built in the BDDC preconditioner, it is embedded within the FETI-DP operator. For deterministic systems, it has already been demonstrated that the condition number and thus the parallel performance of BDDC and FETI-DP are quite similar [35,41–45]. Indeed, the preconditioned operators in BDDC and its counterpart in FETI-DP have the same algebraic structure with identical eigenvalues except possibly for zero and one [41]. This fact indicates that the convergence properties of both BDDC and FETI-DP are similar for the same primal constraints [42]. It is therefore natural to ask whether the similarity of BDDC and FETI-DP extends to stochastic systems. We address this question in part using numerical experiments in this paper.

Within a general framework of iterative substructuring techniques for SPDEs [1,2,28–33], we formulate a probabilistic version of FETI-DP for the iterative solution of the deterministic linear system in the SSFEM [1,2,46]. In the dual-primal method for stochastic systems, the physical domain is decomposed into a number of non-overlapping subdomains. The polynomial chaos expansion is used to represent the uncertain system parameters and solution processes. For each subdomain, the solution vector are divided into the interior and interface unknowns [36,37]. The interface unknowns are further split into the sets of corner and remaining variables. In each iteration of the preconditioned conjugate gradient method (PCGM) for the probabilistic dual interface system, the global assembly for the chaos coefficients of the solution process at the corner nodes is performed to strictly enforce the continuity conditions.In addition, Lagrange multipliers are utilized to weakly satisfy the continuity of the polynomial chaos coefficients on the remaining nodes of the interface. Consequently, a probabilistic coarse problem is embedded in the operator of the dual interface system of FETI-DP. The introduction of the coarse problem in the probabilistic framework leads to a scalable performance of FETI-DP solver for large-scale stochastic systems.

The convergence properties, numerical and parallel scalabilities of both the probabilistic BDDC and FETI-DP are contrasted for two-dimensional flow through porous media and linear elasticity problems with spatially varying system parameters modeled as non-Gaussian stochastic processes. The weak and strong scalabilities of the algorithms are investigated on a Linux cluster consisting of 22 nodes with 2 Quad-Core 3.0 GHz Intel Xeon processors and 32 GB of memory per node with InfiniBand interconnect. The algorithms are implemented using MPI [47] and PETSc [48] parallel libraries. The finite element mesh is decomposed using METIS [49] graph partitioning tool.

In this paper, we present a brief review of one- and two-level primal domain decomposition approaches for SPDEs in Sections 2–4. For a comprehensive review of the mathematical framework of these algorithms, we refer to [46]. The dual-primal approach for SPDEs is introduced in Section 5. Preconditioners for the probabilistic FETI-DP method are introduced in Section 6. We show a connection between the primal and dual-primal approaches for uncertainty quantification in Section 7. The parallel implementation of the proposed probabilistic FETI-DP algorithm is outlined in Section 8. The numerical results are presented in Section 9 and some conclusions are drawn in Section 10.

2. Schur complement system of the stochastic problems

The finite element approximation of an elliptic SPDE leads to the following linear system (e.g. [3,17])

$$\mathbf{A}(\theta)\mathbf{u}(\theta) = \mathbf{f}, \tag{1}$$

where $\mathbf{A}(\theta)$ is the stiffness matrix with random coefficients, $\mathbf{u}(\theta)$ denotes the stochastic response vector and $\mathbf{f}$ represents the external force vector. For large-scale systems, we exploit iterative substructuring techniques to solve Eq. (1) as detailed in [1,2,28–33]. The iterative substructuring algorithm involves partitioning the computational domain Ω into n_s subdomains, $\Omega = \bigcup_{s=1}^{n_s}\Omega_s$, with the interface boundary $\Gamma = \bigcup_{s=1}^{n_s}\Gamma_s$ where $\Gamma_s = \partial\Omega_s \setminus \partial\Omega$. Using this decomposition, the subdomain equilibrium system can be expressed as [1,2,28–33]

$$\begin{bmatrix} \mathbf{A}^s_{II}(\theta) & \mathbf{A}^s_{I\Gamma}(\theta) \\ \mathbf{A}^s_{\Gamma I}(\theta) & \mathbf{A}^s_{\Gamma\Gamma}(\theta) \end{bmatrix} \begin{Bmatrix} \mathbf{u}^s_I(\theta) \\ \mathbf{u}^s_\Gamma(\theta) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^s_I \\ \mathbf{f}^s_\Gamma \end{Bmatrix}, \tag{2}$$

where $\mathbf{u}^s_I(\theta)$ corresponds to the interior unknowns of the subdomain Ω_s and $\mathbf{u}^s_\Gamma(\theta)$ denotes the interface unknowns shared among adjacent subdomains as shown schematically in Fig. (1).

Representing the stochastic system parameters using PCE, leads to the following system for the subdomain Ω_s as

$$\sum_{i=0}^{L}\Psi_i(\theta)\begin{bmatrix} \mathbf{A}^s_{II,i} & \mathbf{A}^s_{I\Gamma,i} \\ \mathbf{A}^s_{\Gamma I,i} & \mathbf{A}^s_{\Gamma\Gamma,i} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^s_I(\theta) \\ \mathbf{u}^s_\Gamma(\theta) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^s_I \\ \mathbf{f}^s_\Gamma \end{Bmatrix}, \tag{3}$$

where $\Psi_i(\theta)$ is a set of orthogonal multi-dimensional Hermite polynomials. The orthogonality property is defined as [3,15,16]

$$\langle \Psi_i(\theta)\Psi_j(\theta)\rangle = \langle \Psi_i^2(\theta)\rangle\delta_{ij}, \tag{4}$$

where δ_{ij} is the Kronecker delta and $\langle\cdot\rangle$ denotes the expectation operator.