Comput. Methods Appl. Mech. Engrg. 265 (2013) 213–225

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

An explicit solution for implicit time stepping in multiplicative finite strain viscoelasticity

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ARTICLE INFO

Article history:
Received 11 April 2013
Received in revised form 17 June 2013
Accepted 8 July 2013
Available online 18 July 2013

Keywords:
Viscoelasticity
Maxwell fluid
Finite strains
Implicit time stepping
Integration algorithm
Euler-backward method

ABSTRACT

We consider the numerical treatment of one of the most popular finite strain models of the viscoelastic Maxwell body. This model is based on the multiplicative decomposition of the deformation gradient, combined with Neo-Hookean hyperelastic relations between stresses and elastic strains. The evolution equation is six dimensional and describes an incompressible flow such that the volume changes are purely elastic. For the corresponding local initial value problem, a fully implicit integration procedure is considered, and a simple explicit update formula is derived. Thus, no local iterative procedure is required, which makes the numerical scheme more robust and efficient. The resulting integration algorithm is unconditionally stable and first order accurate. The incompressibility constraint of the inelastic flow is exactly preserved. A rigorous proof of the symmetry of the consistent tangent operator is provided. Moreover, some properties of the numerical solution, like invariance under the change of the reference configuration and positive energy dissipation within a time step, are discussed. Numerical tests show that, in terms of accuracy, the proposed integration algorithm is equivalent to the classical implicit scheme based on the exponential mapping. Finally, in order to check the stability of the algorithm numerically, a representative initial boundary value problem involving finite viscoelastic deformations is considered. A FEM solution of the representative problem using MSC.MARC is presented.

1. Introduction

Among idealized models of linear viscoelasticity, the so-called Maxwell fluid (MF) is commonly encountered in material modeling [52,17,2]. The one-dimensional rheological interpretation of this model is shown in Fig. 1(a).[1] A series of Maxwell elements connected in parallel [66] can be utilized to represent viscoelastic properties of polymers (Fig. 1(b)). In that case, the stresses acting in the Maxwell elements can be associated with overstresses [17]. Next, a slightly modified Maxwell element can be adopted to capture the nonlinear kinematic hardening in metals (Fig. 1(c)). In that case, the corresponding Maxwell stresses are interpreted as backstresses [39,3,65,7,56]. Moreover, within some phenomenological approaches to metal plasticity, the distortional hardening in metals can be captured using the modified MF [57,59]. Other groups of materials like shape memory alloys [19,21] and biological tissues [8] can be modeled using MF. Some applications of MF to finite deformations of geological structures [60,48] and to fluid mechanics [1] are reported in the literature as well.

In the finite strain range, numerous constitutive models of the MF exist (see, among others, [37,28,33,47,45,46,24,4,50,1,17,18 ,35]). Different variants were compared through numerical tests in [10,34]. In this paper we consider one of the most popular models of the MF. The corresponding constitutive equations are summarized in Section 2 of this work. The model under consideration is a special case of the finite strain viscoplasticity model proposed by Simo and Miehe [61], and it has the same structure as the well known model of associative elastoplasticity considered by Simo [62]. These models were developed within the framework of multiplicative inelasticity in combination with hyperelastic constitutive relations. The corresponding inelastic flow rule[2] is six dimensional since the inelastic spin plays no role due to elastic isotropy. A version of the MF which is equivalent to the version of Simo and Miehe was considered later in material (Lagrangian) description by Lion [38]. This Lagrangian formulation was adopted in [40,19,13,53,55–58]. The spatial (Eulerian) constitutive equations proposed by Simo and Miehe were utilized later in the comprehensive study by Reese and Govindjee [50], and by many others (see, for instance, [25,43,48,31,16,23,49,36]).

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[1] A two-dimensional rheological model of the Maxwell fluid and its modifications can be found in [57,59].

[2] In this paper, the evolution equation is referred to as "inelastic flow rule" in order to stress that the model is a special case of a viscoplastic model.

0045-7825/$ - see front matter
http://dx.doi.org/10.1016/j.cma.2013.07.004

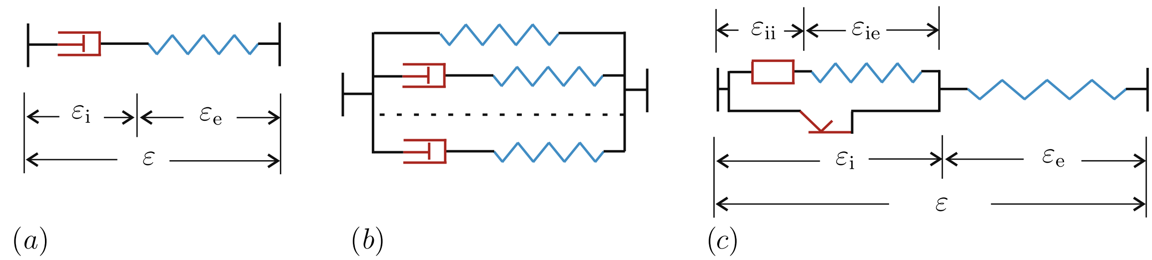

Fig. 1. (a) A one-dimensional Maxwell body consists of an elastic spring (Hooke body) coupled in series with a viscous dashpot (Newton body), (b) Generalized Maxwell body, also known as Wiechert model (or Zener model in a special case) used for description of viscoelastic properties, (c) modified Schwedoff model used to represent nonlinear kinematic hardening.

In the modern literature on numerical mechanics, much attention is paid to general procedures which can be implemented to different types of constitutive relations in a straightforward manner. Because of their generality, such procedures are not always efficient being compared to algorithms which make use of the special structure of the underlying constitutive equations. Due to the high prevalence of the Simo and Miehe version of the Maxwell fluid in material modeling, efficient and robust numerical integration of the underlying evolution equations is a challenging task. *The main purpose of this paper is to report a new, simple, and efficient numerical procedure for this model.*

Since the corresponding initial value problem is typically stiff, implicit time stepping methods should be implemented. For the Simo and Miehe version of the MF, a discretized problem can be obtained using the operator split technique in combination with exponential mapping and formulation in principal axes as described in [62,50] (Eulerian approach). Alternatively, the evolution equation formulated on the reference configuration can be discretized as described in [11] (Lagrangian approach). In both cases, a system of nonlinear algebraic equations is obtained, and a local iterative procedure is usually implemented to resolve the resulting nonlinear problem (see, among others, [50,44,13,14,48,20,31,54,65,16,23,49,36]). Obviously, such iterative procedures can slow down the entire FEM simulation. This problem may become especially important if globally explicit FEM is considered.[3] This publication is dealing with first-order accurate methods only. For the discussion concerning the application of higher order methods, the reader is referred to [51,15,5,6].

In order to speed up the FEM computations, much attention was paid to the construction of closed form solutions for implicit schemes. For instance, a simplified flow rule under the assumption of small elastic strains was considered by Simo and Miehe [61] in order to get an explicit update formula for the local implicit time stepping procedure. For the same reason, another simplification of the flow rule in case of small elastic strains was considered by Reese and Govondjee [50]. This simplified version was implemented later in [27]. Unfortunately, the simplifying assumption of small elastic strains is not valid for many materials like plastics, rubber, biological tissues etc. Moreover, if the modified Maxwell body is used to capture nonlinear kinematic hardening in metals, a general finite strain version of the model must be utilized as well.[4] Another approach to closed form solution is based on special assumptions concerning the energy storage. In particular, a quadratic logarithmic strain energy (so-called Hencky strain energy) can be assumed in order to simplify the numerical treatment of the material model [41]. Unfortunately, this assumption would yield unrealistic results in case of large elastic strains. Thus, again, the applicability area is limited to moderate elastic strains. In this work, a simple explicit update formula is presented for the original finite strain version with Neo-Hookean potential. Interestingly, this explicit solution for the general case is even more compact and simple than the solutions presented in [61,50] for the special case of small elastic strains or the solution in [41] for quadratic logarithmic strain energy. For the new method, the computational effort per single time step is even smaller than the effort required within the explicit time stepping.

The inelastic flow is assumed to be incompressible, and the algorithm presented in this work preserves this incompressibility constraint. A classical model of finite strain viscoplasticity which contains the Simo and Miehe version of the MF was considered in [55]. As it was shown in [55], the exact solution to the initial value problem is exponentially stable with respect to small perturbations of the initial data, if the incompressibility constraint is not violated. For such material models, the numerical schemes which exactly preserve the incompressibility are advantageous due to the *suppressed error accumulation* [55]. This theoretical result is confirmed by numerical tests presented in the current paper.

Dealing with the constitutive equations written in Lagrangian form, it can be shown that they are invariant under isochoric changes of the reference configuration [58]. The same invariance property can be formulated for the numerical solution as well. Obviously, the numerical algorithms which exactly retain this invariance property are advantageous. In this work, it is proved that the advocated algorithm retains the invariance of the solution.

We close this introduction with a few words regarding notation. Throughout this article, bold-faced symbols denote first- and second-rank tensors in $\mathbb{R}^3$. A coordinate-free tensor formalism is used in this work [26,54]. In this work, $\mathbf{1}$ stands for the second-rank identity tensor. The deviatoric part of a tensor is defined as $\mathbf{A}^{\mathrm{D}} := \mathbf{A} - \frac{1}{3}\mathrm{tr}(\mathbf{A})\mathbf{1}$, where $\mathrm{tr}(\mathbf{A})$ stands for the trace. The material time derivative is denoted by dot: $\frac{d}{dt}\mathbf{A} = \dot{\mathbf{A}}$. The overline $\overline{(\cdot)}$ denotes the unimodular part of a tensor such that

$$\overline{\mathbf{A}} = (\det \mathbf{A})^{-1/3}\mathbf{A}. \tag{1}$$

The inverse of transposed tensor is denoted by $\mathbf{A}^{-\mathrm{T}}$. The positive definiteness of a tensor $\mathbf{A}$ is symbolically denoted by $\mathbf{A} > 0$.

2. System of constitutive equations

2.1. Lagrangian formulation

Let us consider a finite strain model of Maxwell fluid. This model is covered as a special case by the viscoplasticity model presented by Simo and Miehe [61]. The Lagrangian formulation of the model follows the presentation of Lion [38]. Let $\mathbf{F}$ be the deformation gradient from the local reference configuration $\tilde{\mathscr{K}}$ to the current configuration $\mathscr{K}$. We start with the multiplicative

[3] In the case of explicit FEM, the evaluation of the material routine at each point of Gauss integration constitutes the major part of the overall computational effort.

[4] In fact, although the elastic strains in metals are typically small ($\varepsilon_{\mathrm{e}} \to 0$ in Fig. 1(c)), the conservative part ($\varepsilon_{\mathrm{ie}}$ in Fig. 1(c)) of the inelastic strain may become finite.