Comput. Methods Appl. Mech. Engrg. 264 (2013) 12–22

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Analysis of a viscoelastic contact problem with multivalued normal compliance and unilateral constraint

Mircea Sofonea [a], Weimin Han [b,*], Mikael Barboteu [a]

[a] Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France
[b] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

ARTICLE INFO

Article history:
Received 18 January 2013
Accepted 10 May 2013
Available online 25 May 2013

Keywords:
Frictionless contact
Normal compliance
Unilateral constraint
History-dependent variational inequality
Error estimates
Numerical simulations

ABSTRACT

We consider a mathematical model which describes the quasistatic contact between a viscoelastic body and a foundation. The material's behavior is modeled with a constitutive law with long memory. The contact is frictionless and is modeled with a multivalued normal compliance condition and unilateral constraint. We present the classical formulation of the problem, list the assumptions on the data and derive a variational formulation of the model. Then we prove its unique solvability. The proof is based on arguments of history-dependent quasivariational inequalities. We also study the dependence of the solution with respect to the data and prove a convergence result. Further, we introduce a fully discrete scheme to solve the problem numerically. Under certain solution regularity assumptions, we derive an optimal order error estimate. Finally, we provide numerical validations both for the convergence and the error estimate results, in the study of a two-dimensional test problem.

1. Introduction

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general Mathematical Theory of Contact Mechanics is currently emerging. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws, i.e. materials, varied geometries and different contact conditions. Comprehensive references on the topic include the monographs [5–7,11,18,20–22,24]. The state of the art in the field can be found in the proceedings [17,25,27] as well. The aim of this paper is to study a frictionless contact problem for rate-type viscoplastic materials within the framework of the Mathematical Theory of Contact Mechanics. We model the material's behavior with a constitutive law of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain. Here $\mathcal{A}$ is the elasticity operator, allowed to be nonlinear, and $\mathcal{B}$ represents the relaxation operator, assumed to be linear.

Quasistatic contact problems for materials following the law (1.1) can be found in [24] and the references therein. There, the contact was assumed to be frictionless and was modeled with normal compliance and unilateral constraint; the unique weak solvability of the corresponding problems was proved by using arguments of history-dependent variational inequalities. The normal compliance contact condition was first introduced in [19] and since then used in many publications, see, e.g., [11–13,16] and references therein. The term *normal compliance* was first introduced in [12,13]. The current paper has three traits of novelties that we describe in what follows. First, the model we consider involves a contact condition with multivalued normal compliance and unilateral constraint. This condition takes into account both the deformability and the rigidity of the foundation. Second, we provide the numerical analysis of the problem, including error estimates for fully discrete scheme. Last, we present numerical simulations which validate our theoretical results. The rest of the paper is structured as follows. In Section 2 we present the notation as well as some preliminary material. In Section 3 we describe the model of the contact process. In Section 4 we list the assumptions on the data and derive the variational formulation of the problem. Then we state and prove an existence and uniqueness result, Theorem 4.1. In Section 5 we state and prove a convergence result, Theorem 5.1, on the continuous dependence of the solution with respect to the data. In Section 6 we introduce a fully discrete scheme to solve the problem numerically. Under certain solution regularity assumptions, we derive an optimal order error estimate.

* Corresponding author. Tel.: +1 319 335 0770.
E-mail address: weimin-han@uiowa.edu (W. Han).

0045-7825/$ - see front matter
http://dx.doi.org/10.1016/j.cma.2013.05.006

Finally, in Section 7 we present numerical simulation results on a two-dimensional example.

2. Notations and preliminaries

We use $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ for the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. For a given $r \in \mathbb{R}$ we denote by r^+ for its positive part, i.e. $r^+ = \max\{r, 0\}$. Let Ω be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that $\text{meas}\,(\Gamma_1) > 0$. We use $\boldsymbol{x} = (x_i)$ for a generic point in $\Omega \cup \Gamma$ and denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal on Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

We use standard notation for Lebesgue and Sobolev spaces in Ω and on Γ. Let

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}, \quad Q = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d \times d} : \tau_{ij} = \tau_{ji}\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from Korn's inequality due to the assumption $\text{meas}(\Gamma_1) > 0$.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 depending on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leqslant c_0 \|\boldsymbol{v}\|_V \quad \forall \boldsymbol{v} \in V. \tag{2.1}$$

For a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that with the divergence operator defined by the equality $\text{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \text{Div}\,\boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall \boldsymbol{v} \in V. \tag{2.2}$$

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) : \ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \leqslant i, j, k, l \leqslant d\}$$

which is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \sum_{1 \leqslant i,j,k,l \leqslant d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

A simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \leqslant \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \quad \forall \mathcal{E} \in \mathbf{Q}_\infty, \ \boldsymbol{\tau} \in Q. \tag{2.3}$$

For each Banach space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Details can be found in [4,15], for instance. Here we only recall that the convergence of a sequence $(v_k)_k$ to an element v, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} v_k \to v \text{ in } C(\mathbb{R}_+; X) \text{ as } k \to \infty \text{ if and only if} \\ \max\limits_{t \in [0,n]} \|v_k(t) - v(t)\|_X \to 0 \text{ as } k \to \infty, \text{ for all } n \in \mathbb{N}^*. \end{cases} \tag{2.4}$$

Consider now a real Hilbert space X with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$. Let K be a subset of X and consider operators $A : K \to X$, $\mathcal{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ as well as functions $j : K \to \mathbb{R}, f : \mathbb{R}_+ \to X$ with the following properties.

$$K \text{ is a nonempty, closed, convex subset of } X. \tag{2.5}$$

$$\begin{cases} \text{(a) There exists } m > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \geqslant m\|u_1 - u_2\|_X^2 \quad \forall u_1, u_2 \in K. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \leqslant M\|u_1 - u_2\|_X \quad \forall u_1, u_2 \in K. \end{cases} \tag{2.6}$$

$$\begin{cases} \text{For every } n \in \mathbb{N}^* \text{ there exists } r_n > 0 \text{ such that} \\ \quad \|(\mathcal{R}u_1)(t) - (\mathcal{R}u_2)(t)\|_X \leqslant r_n \int_0^t \|u_1(s) - u_2(s)\|_X\, ds \\ \quad \forall u_1, u_2 \in C(\mathbb{R}_+; X), \ \forall t \in [0, n]. \end{cases} \tag{2.7}$$

$$\text{The function } j : K \to \mathbb{R} \text{ is convex and lower semicontinuous.} \tag{2.8}$$

$$f \in C(\mathbb{R}_+; X). \tag{2.9}$$

The following result will be used in Section 4 of this paper.

Theorem 2.1 (*Assume (2.5)–(2.9)*). *Then there exists a unique function $u \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$(Au(t), v - u(t))_X + ((\mathcal{R}u)(t), v - u(t))_X + j(v) - j(u(t)) \geqslant (f(t), v - u(t))_X \quad \forall v \in K. \tag{2.10}$$

Theorem 2.1 represents a particular case of a more general result proved in [23]. Following the terminology introduced there, we refer to an operator $\mathcal{R}$ satisfying the condition (2.7) as a *history-dependent operator*. Moreover, (2.10) represents a *history-dependent quasivariational inequality*.

3. The model

The physical setting is as follows. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$ in Ω and of surface tractions of density $\boldsymbol{f}_2$ on Γ_2. It is fixed on Γ_1 and is in frictionless contact on Γ_3 with a deformable obstacle, the so-called foundation. We assume that the contact process is quasistatic and we study it in the interval of time $\mathbb{R}_+ = [0, \infty)$. Then, the classical formulation of the contact problem we consider in this paper is the following.