Comput. Methods Appl. Mech. Engrg. 264 (2013) 108–128

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

High scalable non-overlapping domain decomposition method using a direct method for finite element analysis

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ARTICLE INFO

Article history:
Received 4 August 2012
Received in revised form 3 April 2013
Accepted 13 May 2013
Available online 10 June 2013

Keywords:
FETI
Non-overlapping DDM
FEM
Parallel method

ABSTRACT

In this paper, a novel domain decomposition method is introduced to solve a large finite element model. In this method, the decomposed domains do not overlap and are connected using a simple connective finite element, which influences the nodal point equilibrium between adjacent finite elements. This approach has the advantage that it allows use of a direct method such as Gauss elimination even in a singular problem. The singular stiffness matrices from the floating domain without the Dirichlet boundary conditions are changed into invertible stiffness matrices by assembling the connective elements. Another advantage is that computational time and storage can be reduced by using a banded matrix in the direct solver. In order to describe this proposed method, we first review the FETI method, which is the most popular domain decomposition method. Then the proposed method is introduced with a technical approach in a distributed computer system. Finally, the high scalability and computational efficiency of the proposed method are verified by comparing with the traditional FETI method for 2D and 3D finite element examples which have floating subdomains. In the result, we demonstrate high scalability for a large finite element model.

1. Introduction

The finite element method (FEM) is one of the most popular numerical methods in engineering. Several FEM methodologies have been developed over the past decades. Usually, two main factors are considered in FEM: numerical accuracy and efficiency. In order to obtain more accurate results, computational inefficiency is increased by using smaller mesh size or more meshes. This is an important consideration in a large system with many equations because fast processors and vast storage are needed. Therefore, appropriate selection of mesh size and element number is primary factor influencing performance and efficiency during computation.

Direct methods in FEM are often used for solving sparse linear equations in a static or implicit dynamic analysis. Gauss elimination is currently the most effective direct method. It calculates accurate results rapidly for inversing stiffness matrices in small-scale systems with a small number of degrees of freedom. However, for solving a large system with many degrees of freedom, this general direct method has limitations. Computational speed declines and a vast storage is needed even though stiffness matrices in FEM are symmetrical, positive definite and banded. The frontal solution variant of Gauss elimination is proposed to overcome this drawback [1]. When stiffness matrices are assembled, an order of equation is eliminated at the same time. Therefore, the calculation is very fast and requires little memory, but the order of assembly ought to be carefully determined. This method has been considered successful by many researchers; moreover, many commercial programs use it at present. Later, the extended frontal method, the so-called multifrontal method is proposed [2], and it is possible to use parallel computing. In this method, some different fronts are used, and better efficiency and stability can be obtained for numerous problems. Recently, research based on this frontal method has been performed with parallel algorithms [3–5].

In addition to a frontal solution based on the direct method, the parallel method is also an effective method of solving large sparse linear systems in FEM. The domain decomposition method (DDM) is based on an iterative method and has been adapted to solve physically different problems like fluid–structure interactions, multiphase issues, and multiphysical structures. The mathematical approach was first presented by Schwarz [6] to solve partial differential equations. The basic idea is that the original domain is first split into subdomains, and each subdomain is computed by independent solutions. In a computational approach, this method is generally used with an iterative solution to solve a boundary value problem between adjacent subdomains. The DDM is mainly classified into overlapping methods and non-overlapping methods. In overlapping methods, subdomains on the interface overlap, and unknown variables at the interface are solved by iterative methods

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http://dx.doi.org/10.1016/j.cma.2013.05.013

with prescribed Dirichlet boundary conditions [7–9]. These methods are based on the Additive Schwarz method. On the other hand, in non-overlapping methods, interconnecting parameters like Lagrange multipliers enforce continuity along the interface through the use of unknown variables. The finite element tearing and interconnect (FETI) method is a very popular non-overlapping DDM [10]. In this method, the Lagrange multipliers represented by interconnecting forces are prescribed into interface nodes on each subdomain, and they are calculated with compatibility conditions in between interface nodes. The FETI method is high scalable because CPUs can independently handle computations on different subdomains in most steps of procedure. However, this method has a severe drawback that numerical difficulty can occur when the subdomain is in a floating situation without Dirichlet boundary conditions; namely, when the stiffness matrix on floating domain is singular. For FETI, both the pseudo-inverse matrix and the kernel of stiffness matrix using factorization or geometric approach are needed to overcome this problem [11,12]. Alternatively, the dual-primal finite element tearing and interconnecting method (FETI-DP) is proposed [13], which has been extended by many researches [14–20]. The basic concept of the method is that some interface nodes are imposed by continuity constraints to represent primal variables, and other interface nodes are forced by Lagrange multipliers called dual variables. This brilliant approach makes it possible to invert a stiffness matrix for floating subdomains. In addition, this DDM is used as an iterative method with preconditioners. Therefore, when a large system is decomposed, appropriate pre-conditioners are needed to obtain better convergence, but numerical inefficiency can sometimes occur. In order to avoid pre-conditioners with the iterative method, a few DDM using direct methods have been attempted [21–23].

In this paper, we propose a simple non-overlapping DDM using a direct method. As mentioned above, a direct method is more competent than an iterative method for simple models. Therefore, for large systems, a direct method can be effectively performed with DDM if all small subdomains are prescribed by Dirichlet boundary conditions. Moreover, it is not necessary to handle pseudo-inverse matrix and the kernel of stiffness matrix. This approach provides high scalability with parallel analysis. We attempt to combine the element on the subdomain with an interface element, as prescribed by the Dirichlet boundary condition, in order to create an invertible stiffness matrix. For this, the FETI method is reviewed, and new non-overlapping DDM is proposed. Then, a parallel algorithm for a new method is presented in detail. Finally, high scalability of the proposed method is verified by comparing with FETI method.

2. The finite element tearing and interconnect (FETI) method

The FETI, which is a popular non-overlapping DDM based on iterative solution, is reviewed in this section. In FEM, a system of equations for static solid analysis can be written as

$$\mathbf{K}\mathbf{u} = \mathbf{f} \quad \text{on } \Omega \tag{1}$$

in which $\mathbf{K}$ is the stiffness matrix, which is a symmetric positive definite banded matrix, $\mathbf{u}$ is the displacement vector, $\mathbf{f}$ is the force vector and Ω is the computational domain. When this computational domain Ω is divided into subdomain Ω_i where subscript i is denoted by the ith subdomain, the system Eq. (1) can be defined as follows

$$\mathbf{K}_i\mathbf{u}_i = \mathbf{f}_i - \mathbf{B}_i^T\boldsymbol{\lambda} \quad \text{on } \Omega_i, \quad i = 1,\ldots,N \tag{2}$$

$$\sum_{i=1}^{N}\mathbf{B}_i\mathbf{u}_i = 0 \tag{3}$$

where $\boldsymbol{\lambda}$ is a Lagrange multiplier, N is the number of subdomains, and $\mathbf{B}$ is the connectivity matrix composed with signed Boolean entries.

Assuming that the computational domain is decomposed into a subdomain Ω_1 prescribed by a boundary condition and floating subdomains Ω_{i+1} as in Fig. 1, displacements on each subdomain can be defined by Eq. (2) as follows

$$\mathbf{u}_1 = \mathbf{K}_1^{-1}(\mathbf{f}_1 - \mathbf{B}_1^T\boldsymbol{\lambda}) \tag{4}$$

$$\mathbf{u}_{i+1} = \mathbf{K}_{i+1}^{+}(\mathbf{f}_{i+1} - \mathbf{B}_{i+1}^T\boldsymbol{\lambda}) + \mathbf{R}_{i+1}\boldsymbol{\alpha} \tag{5}$$

where $\mathbf{K}_{i+1}^{+}$ is a pseudo-inverse of $\mathbf{K}_{i+1}$, $\mathbf{R}_{i+1}$ represents the rigid body modes of subdomain Ω_{i+1} which is a basis of the null space of $\mathbf{K}_{i+1}$, and α is a linear combination of its modes. There are two approaches to handle the the rigid body modes $\mathbf{R}$: factorization and geometrical approach [11,12]. We introduce a computation through a factorization because that is more liberal for any problems and simple to understand than geometrical method.

Using factorization, the pseudo-inverse matrix $\mathbf{K}_{i+1}^{+}$ can be simply calculated as follows

$$\mathbf{K}_{i+1}^{+} = \begin{bmatrix} \mathbf{K}_{i+1}^{rr} & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

where $\mathbf{K}_{i+1}^{rr}$ is a factorized full rank matrix, and rank is denoted by superscript r. The basis of the null space $\mathbf{R}_{i+1}$ is defined as

$$\mathbf{R}_{i+1} = \begin{bmatrix} -[\mathbf{K}_{i+1}^{rr}]^{-1}\mathbf{K}_{i+1}^{rn} \\ \mathbf{I}^{nn} \end{bmatrix} \tag{7}$$

where a nullity of $\mathbf{K}_{i+1}$ is denoted by superscript n, and $\mathbf{K}_{i+1}^{rn}$ is a factorized matrix at corresponding rows of $\mathbf{K}_{i+1}^{rr}$. $\mathbf{I}^{nn}$ is an identity matrix.

Mathematically the possible singularity of the matrix $\mathbf{K}_{i+1}$ requires a necessary condition for solvability written as

$$\mathbf{R}_{i+1}^T(\mathbf{f}_{i+1} - \mathbf{B}_{i+1}^T\boldsymbol{\lambda}) = 0 \tag{8}$$

which gives us an additional equation in order to solve $\boldsymbol{\lambda}$, α, and $\mathbf{u}$. Substituting Eq. (5) into the compatibility Eq. (3) and combining those with Eq. (8), we can obtain a mixed equation as follows

$$\begin{bmatrix} \mathbf{F}_{\mathrm{I}} & -\mathbf{G} \\ -\mathbf{G}^T & 0 \end{bmatrix}\begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{Bmatrix} = \begin{Bmatrix} \mathbf{d} \\ \mathrm{e} \end{Bmatrix} \tag{9}$$

where

$$\mathbf{F}_{\mathrm{I}} = \mathbf{B}_1\mathbf{K}_1^{-1}\mathbf{B}_1^T + \sum_{i=1}^{N}\mathbf{B}_{i+1}\mathbf{K}_{i+1}^{+}\mathbf{B}_{i+1}^T \tag{10}$$

$$\mathbf{G} = [\mathbf{B}_1\mathbf{R}_1\mathbf{B}_{i+1}\mathbf{R}_{i+1}\ldots\mathbf{B}_N\mathbf{R}_N] \tag{11}$$

$$\mathbf{d} = \mathbf{B}_1\mathbf{K}_1^{-1}\mathbf{f}_1 + \sum_{i=1}^{N}\mathbf{B}_{i+1}\mathbf{K}_{i+1}^{+}\mathbf{f}_{i+1} \tag{12}$$

$$\mathbf{e} = -[\mathbf{R}_1^T\mathbf{f}_1\mathbf{R}_{i+1}^T\mathbf{f}_{i+1}\ldots\mathbf{R}_N^T\mathbf{f}_N] \tag{13}$$

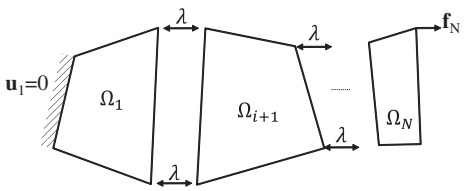

Fig. 1. Two decomposed subdomains Ω_1, Ω_{i+1} to Ω_N of domain Ω.