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# Mixed and Galerkin finite element approximation of flow in a linear viscoelastic porous medium

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ABSTRACT

We propose two fully discrete mixed and Galerkin finite element approximations to a system of equations describing the slow flow of a slightly compressible single phase fluid in a viscoelastic porous medium. One of our schemes is the natural one for the backward Euler time discretization but, due to the viscoelasticity, seems to be stable only for small enough time steps. The other scheme contains a lagged term in the viscous stress and pressure evolution equations and this is enough to prove unconditional stability. For this lagged scheme we prove an optimal order *a priori* error estimate under ideal regularity assumptions and demonstrate the convergence rates by using a model problem with a manufactured solution. The model and numerical scheme that we present are a natural extension to 'poroviscoelasticity' of the poroelasticity equations and scheme studied by Philips and Wheeler in (for example) [Philip Joseph Philips, Mary F.Wheeler, Comput. Geosci. 11 (2007) 145–158] although — importantly — their algorithms and codes would need only minor modifications in order to include the viscous effects. The equations and algorithms presented here have application to oil reservoir simulations and also to the condition of *hydrocephalus* — 'water on the brain'. An illustrative example is given demonstrating that even small viscoelastic effects can produce noticeable differences in long-time response. To the best of our knowledge this is the first time a mixed and Galerkin scheme has been analysed and implemented for viscoelastic porous media.



## 1. Introduction and motivation

In this article we consider an extension to the equations of poroelasticity by modelling the flow of a slightly compressible single phase fluid in a viscoelastic porous medium. The constitutive equations therefore allow for the presence of viscoelastic relaxation effects in the porous media (but not the fluid). Fully discrete numerical schemes are derived based on a lagged and non-lagged backward Euler time stepping method applied to a mixed and Galerkin finite element spatial discretization. We show that the lagged scheme is unconditionally stable and give an optimal *a priori* error bound for it. Furthermore, this scheme is practical and useful in the sense that it can be easily implemented in existing poroelasticity software because the coupling between the viscous stresses and pressures and the elasticity and flow equations is 'lagged' by one time step. The required additional coding therefore takes the form of extra 'right hand side loads' together with some updating subroutines for the viscoelastic *internal variables*, but the solver and assembly engines remain intact. This idea of lagging has been used before for nonlinearly viscoelastic diffusion problems in [3,24] but, of course, is not new. Lagging in numerical schemes is discussed more widely by Lowrie in [14].

This work was originally motivated by geomechanics applications but during its development we have become aware of its potential relevance to the modelling of cerebrospinal fluid flow and its relation to the condition of *hydrocephalus*. To the best of our knowledge this is the first time a mixed and Galerkin scheme has been analysed and implemented for viscoelastic porous media.

### 1.1. Geomechanics

Reservoir simulators are built by computationally solving partial differential equations that employ Darcy's law to approximate the flow through porous media. The oil reservoirs can be anywhere between 300 m to 10 km below the earth's surface in the litho-

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sphere. At the simplest level of modelling the lithosphere (the porous medium) can be considered as pefectly rigid but, in practice, it is more accurately modelled as being either elastic or viscoelastic as in, for example, [2, Chap. 2, 18,6,31]. The point made by Lakes in [12, Section 7.4.1] is that although at room temperatures rock is not in general a 'lossy' medium, at the elevated temperatures in the Earth's interior the viscoelastic loss tangents can be significant. Also in [12, Section 8.3.1], an explanation of viscoelastic behaviour of porous media even at cooler temperatures is given based on the time and frequency dependent drag forces from the stress-induced fluid flow.

Recently Philips and Wheeler in [20–22] and then Wheeler and Gai in [32] described, discretized and analysed a poroelasticity model in which the porous rock was allowed to behave linear elastically (see for example [7,5]). Rohan et al. in [25] then followed by using homogenisation techniques to extend that poroelasticity model by including linear viscoelastic effects. Under the assumption of slow fluid flow, that model — considered below — is able to simulate relaxation and creep behaviour, as well capture damping and frequency dependent behaviour (see the interesting article [4] for an idea of the importance of viscoelastic damping in geology).

### 1.2. Cerebrospinal fluid (CSF) flow

Our original connection to this potential application came through exposure to the work that now appears in [9]. Here the flow of CSF through the ventricles of an elastic-sponge-like brain is modelled using essentially the same equations of poroelasticity as touched on above. The work in [9] follows on from the developments presented in [28,34] and is related to the studies in [26,27,33]. The last authors note that brain tissue is in general viscoelastic as described in, for example, [30,17,29,19], [12, Section 7.5.7] and this provides the connection to the work presented in [25] and below.

We should also mention that the model in [9] allows for nonlinear compression-dependent effects, and also that [35] extends the model to finite strain hyperelasticity.

### 1.3. Poro-visco-elasticity

Although the idea of viscoelastic porous media modelling and numerics is not new (see also [8] and the comprehensive [15] as well as the those above) we believe that this paper is the first to present it in a mixed and Galerkin framework.

The viscoelasticity of the porous media is introduced into the poroelasticity model by using a stress relaxation ODE (ordinary differential equation) for an 'internal stress variable' rather than using the equivalent (when a Prony series relaxation function is assumed) notion of a 'hereditary integral'. This extension of Hooke's law to linear viscoelasticity is classical and very well documented in the literature (see, for example, [11,10]). What is not so obvious is how the viscoelasticity of the skeleton influences the flow equation for pressure. To reveal this mechanism Rohan et al. in [25] used homogenization arguments to derive the governing equations that appear below.

Although for the reasons touched on earlier this viscoelastic porous media model is useful in its own right, in another respect it serves (at least mathematically) as a starting point for adding other forms of internal variable equations. These can represent more complicated behaviour such as, for example, plasticity as formulated in [1]. We hope to return to these extensions at a later time as well as to other important topics such as the *thermoporoelasticity* model described in [13].

We now move on to describe the model with which we shall be concerned. This will be followed in Section 3 with the numerical scheme; in Section 4 with a derivation of error bounds; in Section 5 with an illustration of these bounds and in Section 6 with a more practically-oriented demonstration of the model. We finish in Section 7 with some concluding remarks.

In isotropic linear elasticity theory in $\mathbb{R}^d$ the symmetric stress tensor, $\underline{\boldsymbol{\sigma}} = (\sigma_{ij})_{i,j=1}^d$ is related to the strain tensor, $\underline{\boldsymbol{\varepsilon}} = (\varepsilon_{ij})_{i,j=1}^d$ through the constitutive law,

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}} \quad \text{or} \quad \sigma_{ij} = \lambda\varepsilon_{kk}(\boldsymbol{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u}),$$

where $\varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}(u_{i,j} + u_{j,i})$ and with $\boldsymbol{u} = (u_i)_{i=1}^d$ the displacement and $\lambda, \mu$ the Lamé constants. Unless explicitly stated otherwise the summation convention is in force throughout and we usually suppress $\boldsymbol{x}$ dependence to enhance readability. Note that $\underline{\mathbf{D}}$ is positive definite on the symmetric second-order tensors and also that we are writing tensors of order one ('vectors') in bold and tensors of order two or four in bold underline.

The simplest way of including viscoelastic effects such as stress relaxation and creep is to introduce a history functional into the constitutive law (see e.g. [10,11]). For this we introduce the *stress relaxation function* $\varphi(t) = \varphi_0 + \varphi_1 e^{-t/\tau}$, for constants $\varphi_0 > 0, \varphi_1 \geqslant 0$ and $\tau > 0$ such that $\varphi(0) = 1$, and write the stress as,

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) + \int_0^t \dot{\varphi}(t-s)\underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(s))\,ds,$$

where, here and below, the overdot signifies partial differentiation with respect to the (time) argument. It is a fundamental observation that with $\psi_0 = 1/\varphi_0$ and $\psi_1 = \varphi_1/\varphi_0$ this relationship can be inverted to give,

$$\underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) = \underline{\boldsymbol{\sigma}}(t) + \int_0^t \dot{\psi}(t-s)\underline{\boldsymbol{\sigma}}(t)\,ds,$$

where $\psi(t) = \psi_0 - \psi_1 e^{-\varphi_0 t/\tau}$ is the *creep function*. Furthermore, noting that $\dot{\varphi}(t-s) = -\tau^{-1}\varphi_1 \exp(-(t-s)/\tau)$ we define the *internal stress variable*

$$\underline{\boldsymbol{\sigma}}^*(t) := \int_0^t \frac{\varphi_1}{\tau} e^{-(t-s)/\tau}\underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(s))\,ds \tag{1}$$

and get

$$\tau\dot{\underline{\boldsymbol{\sigma}}}^* + \underline{\boldsymbol{\sigma}}^* = \varphi_1\underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) \quad \text{subject to} \quad \underline{\boldsymbol{\sigma}}^*(0) = \underline{\mathbf{0}}.$$

With this we can write $\underline{\boldsymbol{\sigma}}(t) = \underline{\mathbf{D}}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)) - \underline{\boldsymbol{\sigma}}^*(t)$ and thereby remove the explicit appearance of the displacement history.

Now letting $p$ denote the pressure field and assuming that $p$ and $\boldsymbol{u}$ are zero at $t = 0$ we appeal to the simplest form of the model presented by Rohan et al. in [25] and, on borrowing terminology from poroelasticity, find that the *total stress*, $\tilde{\sigma}_{ij} := \sigma_{ij} - \alpha\delta_{ij}p$, is given by,

$$\tilde{\sigma}_{ij} = \int_0^t \varphi(t-s)D_{ijkl}\frac{\partial}{\partial s}\varepsilon_{kl}(\boldsymbol{u}(s))\,ds - (\beta_{ij} + \phi\delta_{ij})p, \tag{2}$$

where $\beta_{ij} + \phi\delta_{ij}$ are the Biot stress coefficients with $\underline{\boldsymbol{\beta}}$ symmetric and $\phi > 0$ the volume fraction of the fluid part. We will make the simplifying assumption that $\beta_{ij} = \beta\delta_{ij}$ for a positive real number $\beta$ and then after integration by parts we obtain

$$\tilde{\sigma}_{ij} = D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}(t)) - \alpha\delta_{ij}p + \int_0^t \dot{\varphi}(t-s)D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}(s))\,ds \tag{3}$$

for a constant $\alpha = \beta + \phi$.

Again from [25] we have for the pressure equation that

$$\nabla \cdot K\nabla p = (\phi\gamma + \zeta)\dot{p} + \alpha\delta_{ij}\dot{\varepsilon}_{ij} + \zeta\int_0^t \dot{\psi}(t-s)\dot{p}(s)\,ds, \tag{4}$$

where $\gamma > 0$ denotes the fluid's compressibility and $\zeta$ the magnitude of the skeleton's viscoelastic compressibility. We assume a compressible porous medium so that $\zeta > 0$. It is, perhaps, helpful to remark that we are using slightly different notation to that introduced in [25]: in particular, $\zeta$ and $\eta$ (see later) here correspond to $\hat{\mu}$ and $\tilde{\mu}$ there.