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Higher order cell-based multidimensional upwind schemes for flow in porous media on unstructured grids

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ABSTRACT

Standard reservoir simulation schemes employ single-point upstream weighting for convective flux approximation. These schemes introduce both coordinate-line numerical diffusion and cross-wind diffusion into the solution that is grid and geometry dependent.

New locally conservative cell-based multi-dimensional upwind schemes and higher-order cell-based multi-dimensional upwind schemes that reduce both directional and cross-wind diffusion are presented for convective flow approximation. The new higher-order schemes are comprised of two steps; (a) Higher-order approximation that corrects the directional diffusion of the approximation. (b) Truly multi-dimensional upwind approximation, which involves flux approximation using upwind information obtained by upstream tracing along multi-dimensional flow paths. This approximation reduces cross-wind diffusion. Conditions on tracing direction and CFL number lead to a local maximum principle that ensures stable solutions free of spurious oscillations. The schemes are coupled with full-tensor Darcy flux approximations.

Benefits of the resulting schemes are demonstrated for classical convective test cases in reservoir simulation including cases with full tensor permeability fields, where the methods prove to be particularly effective. The test cases involve a range of unstructured grids with variations in orientation and permeability that lead to flow fields that are poorly resolved by standard simulation methods. The higher dimensional formulations are shown to effectively reduce numerical cross-wind diffusion effects, leading to improved resolution of concentration and saturation fronts.

1. Introduction

First order upwind single-point upstream weighting schemes are still commonly employed in reservoir simulation for integrating the essentially hyperbolic components of the system. These schemes rely upon upwind data (defined here by the left or right state) that is determined by wave speed sign, in a local coordinate system aligned with the local grid geometry. As a consequence, directional diffusion is introduced into the solution that is grid and geometry dependent. The effect can be particularly important for cases where flow is across grid co-ordinate lines and is known as cross-wind diffusion [30,4,6,31,32,16,1,33]. By definition, the single-point upstream weighting scheme defines the control volume face flux by using information that flows across the face. However, crucially when selecting this data, while the criterion is based on the sign of the wave speed at the control volume face, the actual data is defined by the nearest neighbor coordinate value. Here wave speed refers to the directional characteristic wave speed or flow speed. In one dimension, this is sufficient to unambiguously define the scheme in terms of the incoming wave direction. However, in higher dimensions the wave direction (or flow direction) can be at an angle, according to the wave velocity vector direction. The deficiency of the standard scheme is its failure to recognize exactly from where the wave or flow velocity is coming and consequently fail to use the real upwind data. In addition, the one dimensional single-point scheme suffers from excessive numerical diffusion along the coordinate directions. Higher order schemes have been developed for reducing this kind of diffusion, here we cite examples of reservoir simulation developments [4,5,14]. Higher dimensional convection schemes continue to be developed for the essentially hyperbolic systems of reservoir simulation. A scheme that uses the correct upwind direction in the time dependent sense is the Corner Transport Upwind (CTU) scheme [6]. Families of first order genuinely multidimensional upwind schemes on structured and unstructured quadrilateral and triangular grids are presented in [7,23,22,21,24,20] for flow in porous media and in [13,18,12] on a Cartesian grid.

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Nomenclature

Upper-case roman

F_{pi}	pth phase Darcy flux over the ith control volume surface increment
F_{Ti}	total Darcy flux over the ith control volume surface increment
M_{pi}	specified pth phase flow rate
V_p	Darcy velocity of phase p
$\mathbf{K}$	permeability tensor
S_p	saturation of phase p
$\mathbf{S}$	phase saturation vector
$\mathbf{S}_R$	right hand state of the pth phase saturation vector
$\mathbf{S}_L$	left hand state of the pth phase saturation vector
N_p	number of phases
N_S	number of surface increments
N_e	number of subfaces attached to the edge e
N_{loc}	number of local cell vertices
$N_{cell(j)}$	number of cells sharing vertex j

Lower-case roman

$\mathbf{n}$	outward normal vector
f_p	fractional flow of phase p
m_p	specified flow rate of phase p
kr_p	pth phase relative permeability
w_p	pth local wave direction (characteristic or flow velocity)

Upper-case greek

Ψ	porosity
Ω_j	jth control volume
Φ	limiter function

Lower-case greek

ϕ	pressure
$(\xi_{e,k}, \eta_{e,k})$	left and right interpolant parameters at subface k attached to edge e
μ_p	pth phase viscosity
ρ_p	pth phase density
τ_j	jth control volume area
$\sigma_l^{(k)}$	local multidimensional coefficient of subface (k) at vertex of local index l
α_j	net cell-wise coefficient of jth saturation S_j

Symbols

min	minimum
max	maximum
∂_i	partial derivative with respect to i
∇	gradient operator

Abbreviations

CVD	control volume distributed
Multi-D	multidimensional
IMPES	implicit pressure explicit saturation
MPFA	multi point flux approximation
CTU	corner transport upwind
LED	local extrema diminishing
DMP	discrete maximum principle

In this paper, a new higher order family of cell-based multidimensional upwind schemes are presented for reservoir simulation on general unstructured grids in two dimensions. The fundamental first order multidimensional upwind schemes are presented first, followed by extension to higher order accuracy. The higher order multi-dimensional convection schemes are coupled with continuous Darcy-flux approximations.

Locally conservative flux-continuous full-tensor finite-volume schemes have been developed for the essentially elliptic component of the reservoir simulation system. These schemes are control-volume distributed (CVD) (also known as multi-point flux approximations MPFA) where flow variables and rock properties are assigned to the control-volumes of the grid and provide a consistent discretisation of the porous medium pressure equation applicable to general geometry and permeability tensors on structured and unstructured grids, see e.g. [28,8,25,15] and references therein for further details of the Darcy flux approximation.

The schemes are formulated and tested on unstructured grids with spatially varying local grid orientation. The benefits of the new schemes are demonstrated for two-phase flow test cases in two dimensions including full tensor velocity fields on structured and unstructured grids. Full-tensor coefficient velocity fields commonly occur in reservoir simulation, due to local orientation of the grid relative to the permeability field and/or grid skewness and distortion which can induce cross-flow effects, which can be enhanced by the local full-tensor effect due to the local grid orientation.

For steep front problems, the single-point upstream weighting scheme suffers from excessive smearing due to both coordinate-line diffusion and cross-wind diffusion that is inherent in the scheme. In contrast, by upwinding in the correct physical wave direction, cross-wind diffusion is minimized and the schemes provide significant enhancement in resolution of discontinuities that travel across the mesh. An important distinction in this work is the introduction of a cell-based locally conservative multidimensional upwind formulation, where the term cell-based refers specifically to tracing that is performed using fluxes that are normal to control-volume subfaces in the cell. As a result, cell-based subface flux tracing is on a finer scale than edge based tracing, which performs tracing using locally assembled fluxes on each cell edge [21]. A further distinction is the extension of the method to both triangle and quadrilateral grid cells, allowing for quite general grid types to be used. In addition cell-wise stability conditions are derived for a linear flux with variable velocity field for both triangle and quadrilateral grid cells. The resulting stability conditions are derived with respect finer scale subcell flux ratios within each triangle or quadrilateral cell, whereas the previously derived conditions are in terms of accumulated edge-based flux conditions [21,20].

Results are free of spurious oscillations in most cases and the new first order multidimensional schemes require a minimal increase in support. The multidimensional schemes are further enhanced by the development of a higher order multidimensional formulation. The net result is a family of higher order multidimensional schemes that reduces both crosswind diffusion and local coordinate line diffusion.

The flow equations are presented in Section 2. Section 3 is devoted to formulation of the vertex centered finite volume methods. Formulation of first order cell-based multidimensional upwind schemes on general unstructured grids is presented in Section 4. The higher Order formulation is presented in Section 5. Section 6 is dedicated to the formulation of cell-wise families of higher order multidimensional schemes. Two-phase flow results are presented in Section 7 that demonstrate the advantages of the new higher dimensional flux-continuous formulation. Conclusions follow in Section 8. The algorithm for the cell based multidimensional formulations is given in an appendix.