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A regularized boundary integral method in potential theory

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ABSTRACT

A global method for solving the boundary integral equation in potential theory is presented. With subtracting and adding back treatment, several auxiliary functions are employed to reduce the singularities of boundary integrals. After that all the integrals can be directly integrated by arbitrary numerical quadrature without any geometric approximation. The advantage of this method is that high accuracy of numerical quadrature can be maintained with minimal computational nodes in boundary integral, since collocation points can be arranged exactly as quadrature points. The proposed auxiliary functions can also smooth the peaks of nearly singular integrals if they exist. In such a case, little extra work is needed to separate singular and nearly singular integrals into different elements. After separation, both kinds of integrals can be partially transformed into line integrals along their boundaries for the three-dimensional case or several simple formulas for the two-dimensional case.

1. Introduction

Boundary integral methods have been widely used in potential theory and elasticity since Jaswon [1], Symm [2], and Hess and Smith [3]. When solving a physical problem by such boundary integrals, the order of unknown dimensions can be reduced by one. These methods have recently been called boundary element methods or panel methods in different societies. In both methods, the boundaries are discretized into local elements or panels which are interpolated only in low-order polynomials. In those locally approximate elements, the boundary integrals are calculated by either numerical quadratures in boundary element methods or analytical formulas in panel methods. Though the accuracy of integration in each local element is excellent, the total accuracy of these methods is limited by their low-order approximation in geometry. Detailed description of these methods can be found in textbooks by Brebbia [4], or Katz and Plotkin [5].

With such low-order shape interpolations, the approximate geometry only results in low-order accuracy from an overall viewpoint, regardless how accurate the methods of integration are used. In order to prevent the loss of total accuracy, the reduction of geometric error from the interpolation function is important. The best choice is to use true geometry without any modeling if possible. In other words, a global element is a better solution. When performing integration in an element, a high-order formula is much more efficient than those low-order ones. So to integrate a global element with a high-order formula is much better than to integrate a bunch of local elements with low-order formulas. If the quadrature nodes are chosen exactly as collocation nodes, a high-order formula reduces the number of nodes required for the same accuracy. Such a global approach was presented in Refs. [6–9]. Their methods solve singular integrals successfully, but when the nearly singular problems occur, the efficiency may drop significantly. In fact, the nearly singular problem does not come from the integral equation itself, but comes with the numerical integration. Without numerical quadrature, there should not be any nearly singular problem. In spite of the reason of its occurrence, many researchers have proposed different schemes, such as special weighted quadratures, mapping methods, and subtraction and addition methods [10,11]. However, all these methods are essentially developed using local elements. Up to now, a global approach remains unavailable.

In this paper, we utilize a few fundamental mathematical identities with several simple functions to eliminate the singularities in Green's third identity. With subtracting and adding back technique, the order of singular integrands in Green's identity can be reduced, so any high-order quadrature formulas can be directly applied with high accuracy maintained. When nearly singularities occur, a remedy to deal with this situation is to separate the boundary into two parts to prevent the collocation points from being involved in both singular and nearly singular integrals at the same time. Using such an approach, one can successfully deal with the singular integrals and smooth the peaks of nearly singular integrals on the boundary successfully. Though a little compensation on computing time is necessary, this method can still maintain its merit of global approximation and high accuracy. Details are discussed in the following sections.

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2. Theory

Assume that ϕ is a harmonic function, and G is the fundamental solution of the Laplace equation. Green's third identity states that

$$\varepsilon(p)\phi(p) = \oint\!\!\!\!\!\!\int_S \left[\phi(q)\frac{\partial G(p,q)}{\partial n_q} - G(p,q)\frac{\partial \phi(q)}{\partial n_q}\right] dS_q, \tag{1}$$

where S is a closed boundary, $\vec{n}_q$ is the outward unit normal at point q, and $\varepsilon(p)$ is a coefficient depending on the location of point p,

$$\varepsilon(p) = \oint\!\!\!\!\!\!\int_S \frac{\partial G(p,q)}{\partial n_q} dS. \tag{2}$$

Usually, this coefficient is 1 when point p is located inside the domain, zero when p is outside the domain, and 1/2 when p lies on the smooth boundary. The Green's function can be expressed as

$$G(p,q) = \begin{cases} -1/4\pi r & 3D \\ \ln r/2\pi & 2D, \end{cases} \tag{3}$$

in which r is the distance between point p and point q. Substituting Eq. (2) into Eq. (1), yields

$$\oint\!\!\!\!\!\!\int_S [\phi(q) - \phi(p)]\frac{\partial G}{\partial n_q} dS_q = \oint\!\!\!\!\!\!\int_S G\frac{\partial \phi}{\partial n_q} dS_q. \tag{4}$$

It has been shown in [6,9] that the integrand on the left-hand side is bounded and the value can be treated as zero in an average sense at $r = 0$ on a smooth region. Obviously, the singular term on the right-hand side is weakly singular only. Because the Green's functions are different in two and three dimensions, the treatment for them is separated in the following sections.

3. Three-dimensional cases

If the boundary S is blunt, no nearly singular integral will occur during computation. In such a case, a simple way for reducing the weak singularity on the right-hand side of Eq. (4) was proposed by Landweber and his coworkers [6,7]. First, a source distribution is found to make an equi-potential surface on the body, and the velocity potential on the surface for this condition is then determined. Then a source distribution and the subtracting and adding back technique are employed to remove the weak singularity. Their method is very successful in handling the problems for blunt bodies, but when a body has a thin surface or multiple bodies with some part of surfaces being very close, this method may fail. The failure comes from the nearly singular behavior in the integrals. The nearly singular integrals cause most numerical integration to lose its accuracy especially in the dipole term. In the present paper, we propose a modified procedure for dividing the whole boundary integral into two parts, one for singular integral and the other for nearly singular integral. Then they are integrated separately and their global accuracy is kept.

In order to reach this goal, the surface integral of dipole term in Green's identity needs to be converted into a line integral along its boundary. Assume that $\vec{r} = (x,y,z)$ is the position vector from point p to point q, and introduce the vector [12,13]

$$\vec{u} = \frac{z}{r(x^2+y^2)}(y, -x, 0) \tag{5}$$

in which r is the length of vector $\vec{r}$. Since

$$\nabla \times \vec{u} = \frac{\vec{r}}{r^3}, \tag{6}$$

and

$$\frac{\partial G}{\partial n_q} = \frac{\vec{r}\cdot\vec{n}_q}{4\pi r^3}, \tag{7}$$

one has

$$4\pi\varepsilon(p) = \oint\!\!\!\!\!\!\int_S (\nabla \times \vec{u})\cdot n_q dS_q, \tag{8}$$

On part of surface S, say S_i, this surface integral can be converted into a line integral by Stokes' theorem as follows:

$$\int\int_{S_i} \frac{\vec{r}\cdot\vec{n}_q}{r^3} dS_q = \int\int_{S_i} (\nabla \times \vec{u})\cdot n_q dS_q = \oint_{C_i} \vec{u}\cdot\vec{dl}, \tag{9}$$

in which c_i is the boundary curve of S_i.

For a blunt body, there is nothing to do on the left-hand side of Eq. (4). However, when the body is a thin object, the nearly singular behavior may accompany the integrand. One simple way for improving this deficiency is to separate the singular and nearly singular functions into different integrals. After the two kinds of integrals are separated, the treatment can be the same, which is expressed as follows:

$$\int\int_{S_i} \phi(q)\frac{\vec{r}\cdot\vec{n}_q}{r^3} dS_q = \int\int_{S_i} [\phi(q) - \phi(P')]\frac{\vec{r}\cdot\vec{n}_q}{r^3} dS + \phi(P')\oint_{C_i} \vec{u}\cdot\vec{dl}. \tag{10}$$

If point p is on the element S_i, which corresponds to the singular integral, then take p' as p. If point p does not lie on the element but just close to it, take point p' as the collocation point on surface S_i closest to point p, which corresponds to the nearly singular one. After this treatment, the integrand of the surface integrals on the right-hand side remains zero in an average sense for the singular point, and its value is also zero for the nearly singular ones. The abrupt change of kernel has been smoothed out after this treatment, so any numerical integration for this modified integral can maintain its accuracy as it should be. As for the line integral in Eq. (10), only when p lies on or very close to the boundary curve c_i, this term may become singular or nearly singular again. In this situation, the numerical integration for the line integral needs an additional consideration. When point p lies on the curve, assume that the parametric form of curve c_i is $(\xi(t), \eta(t), \zeta(t))$ and point p corresponds to $t = 0$, then

$$\vec{r} = (\xi(t), \eta(t), \zeta(t)) - (\xi(0), \eta(0), \zeta(0)), \tag{11}$$

$$\oint_{c_i} \vec{u}\cdot\vec{dl} = \oint_{c_i} \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{y\xi' - x\eta'}{x^2+y^2} dt. \tag{12}$$

When point q approaches point p, corresponding to $t - 0$, the limiting value of the integrand of Eq. (12) is

$$lim_{t\to 0}\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{y\xi' - x\eta'}{x^2+y^2} = \frac{\zeta'(0)}{\sqrt{[\xi'(0)]^2 + [\eta'(0)]^2 + [\zeta'(0)]^2}} \cdot \frac{\sqrt{[\xi'(0)]^2 + [\eta'(0)]^2}}{2} \cdot \frac{\xi''(0)\eta'(0) - \xi'(0)\eta''(0)}{[[\xi'(0)]^2 + [\eta'(0)]^2]^{3/2}}. \tag{13}$$

The first term on the right-hand side of Eq. (13) is the z-directional cosine of the tangent vector at point p. The last term of Eq. (13) corresponds to the curvature of the projection of curve c_i on the x–y plane at point p. If the second derivative of curve c_i exists and the tangent vector at p is not a zero vector, the integrand of Eq. (19) is regular in the limiting sense. Another possible singular condition that may occur in Eq. (12) is the case when $x = y = 0$ but $z \neq 0$. Geometrically speaking, this means that point p does not lie on this curve, but its vertical projection does. Hence, this is not a real singular case, and the limiting value of its integrand is just the second line of Eq. (13). Finally, when point p is very close to the boundary curve but not on it, a narrow peak of integrand will occur indeed. For the sake of accuracy, one should not neglect it in calculation. A few subdivisions are enough in the narrow region.